Optimality of Zero APR on Credit Cards: An Analytical Framework*

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Abstract

Zero APR lending is prevalent in the U.S. credit card market and thus far very little is known about both the scope of this phenomenon and its economic ramifications. Here, we establish the basic stylized facts characterizing the pricing of card debt in the U.S. and develop an analytic framework to study how interest rates on multi-period credit line-like contracts should be set when debt is unsecured and defaultable. We show that according to the basic theory of unsecured lending suitably extended to allow for promotions interest rates on card debt should price in the expected default risk on a period-by-period basis. We conclude that the data presents a puzzle vis-à-vis the basic theory of unsecured lending. We discuss potential extensions of our theory, such as models of time inconsistency in consumption decisions (hyperbolic discounting).

Keywords: zero APR, credit cards, promotional introductory offers, unsecured debt

JEL codes: E21, D91, G20

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1 Introduction

"Zero APR" promotions are the hallmark of the U.S. credit card lending for some time, with billions of such offerings being mailed to consumers annually and billions of dollars in credit card debt being transferred to promotional accounts annually. Yet, to date, very little is known about the impact of zero APR offers on the actual pricing of credit card debt as well as the economic and regulatory ramifications of this activity. Standard models of unsecured lending in macroeconomics simplify the contractual framework by imposing one period zero profit debt contracts and by construction eliminate the possibility of promotions.[1] These models are thus not only inconsistent with the long term nature of credit card contracts but also unable to speak to how it is priced in the data.

Our paper's goal is to fill this gap by, first, documenting the prevalence of promotional pricing of credit card debt, and second, establishing the normative benchmark of how interest rates on multi-period credit line-like contracts should be set when debt is unsecured and defaultable. Based on the analysis of our model, we argue that the prevalence of promotional pricing in the data presents a puzzle vis-à-vis the basic theory of unsecured lending. We then discuss potential extensions of the baseline theory that could explain promotions, such as models of time inconsistency in consumption decisions (hyperbolic discounting). We leave quantitative modeling of promotional lending and its economic ramifications for future research.

Our data comes from the supervisory collection by the Federal Reserve System for the purposes of the Dodd-Frank Act Stress Test (DFAST). It comprises a panel of all general purpose credit card accounts reported by all bank holding companies subject to DFAST, which covers over 70 percent of all credit card accounts and thus provides an aggregate perspective on the nature of the U.S. credit card debt pricing. Our main empirical findings—pertaining to pre-Covid data—can be summarized into five basic facts: 1) About a quarter of general purpose credit card debt has an introductory promotional status at any point in time; in 80 percent of cases featuring zero annual percentage rate (APR) for an introductory period of over a year on average.[2] 2) After the expiration of the promo period, the rate jumps by about 16 percentage points on average, aligning the average

[1]By the standard model of unsecured lending we mean a model in the spirit of that in Athreya (2002), Chatterjee et al. (2007), Livshits et al. (2007), or Livshits et al. (2010).

[2]Annual percentage rate (APR) refers to the yearly interest generated by a sum that credit card borrowers pay. APR is similar to the finance charge but it additionally includes any fees or additional costs associated with the transaction and does not take compounding into account.

APR on the promo account with the average APR on nonpromo accounts. 3) Delinquency rate on promo debt is about average as for the U.S. credit card market as a whole. 4) There is no evidence of any systematic change in the default risk posed by borrowers between promo origination and its expiration. 5) Most debt on promo accounts originates from promotional balance transfers from other cards, with volume of inbound promotional balance transfers roughly matching the flow of expiring promotional debt.

To study these patterns, we extend the canonical theory of unsecured lending by introducing long-lived credit lines with an option of setting an introductory promotional rate.[3] We assume that lender commitment to terms is consistent with the Credit Card Accountability Responsibility and Disclosure (CARD) Act of 2009. Consequently, lenders are ex post prevented from hiking rates on existing debt or slashing credit limits to force early debt repayment, but they can raise limits, lower interest rates, or slash credit limits on the unutilized portions of the credit lines. Borrowers also do not commit to lender and can refinance debt after each period; however, they are subjected to a friction of switching lenders to sustain promotions in equilibrium.[4] Lenders are Bertrand competitors and, in effect, maximize the utility of the borrowers under the requirement of zero profits. The equilibrium in our model implements the underlying constrained optimal allocation.[5]

The main theoretical result of our paper is that the equilibrium contract features interest rates that price in default risk on a period-by-period basis. In particular, the first period introductory rate should reflect default risk that applies to that period, and as long as it is not significantly different from default risk thereafter, it is similar to the reset rate.[6] Intuitively, the result comes about because consumers in the model are rational and understand that lenders must break even in equilibrium. Accordingly, knowing they must pay for defaulting, consumers prefer contracts that make them internalize the default risk in the marginal cost of borrowing. Through the lens of our theory, then, rationalizaing zero APR promotions requires that, upon contract's origination, the expected default

[3] As referenced in footnote 1.

[4] Under free entry (frictionless refinancing), promotions that result in a net loss during the promotional period are not sustainable in equilibrium. To ensure lenders who offer promotions can break even, this friction delays refinancing and allows incumbents to charge the reset rate for some time after the expiration of the promotion.

[5] An allocation that maximizes welfare subject to economy-wide feasibility constraints, which include the form of market incompleteness and the unsecured nature of debt.

[6] This statement applies to a model with an exogenous default risk and no self-selection into borrowing based on default risk. In a model with an endogenous default risk borrowing may be correlated with default risk. However, as we show in the paper, to the extent that borrowers start from low income state, assuming default risk is negatively correlated with income, interest rate schedules will then end up being decreasing on average, reinforcing the conclusion from our baseline model.

risk during the promotional period is low, and that it sharply jumps thereafter. As our stylized facts shows, this is inconsistent with the data because on average there is no change in credit scores between contract origination and promo expiration.

The second part of our paper explores several extensions of the baseline model to examine the robustness of this result and extensions that could rationalize promotions. There, we consider endogenous default, time-varying endogenous default, simultaneous borrowing and saving to enhance consumption in the state of default, and we also discuss the potential of hyperbolic discounting as a candidate behavioral explanation for zero APR pricing (Laibson, 1997). We find that the result is robust to basic extensions of our theory. We also find that the hyperbolic model offers a potential explanation of the patterns seen in the data. In particular, under the naivete formulation, consumers underestimate the importance of the reset rates because they erroneously expect to borrow less in the future. Lenders exploit it by hiking the reset rates and offering promotions in exchange. Under the sophisticated formulation, that mechanism is not operational because consumers are rational, but consumers still favor promotional rates to commit their "future selves" to borrow less—since ex ante they prefer such an outcome for themselves.

While the hyperbolic model can account for promotions, accepting it as the explanation has important regulatory implications. For example, in the naivete case lenders exploit consumer's bounded rationality, which may warrant consumer protection that limits such offerings in the marketplace.[7] In contrast, under the sophisticated formulation, promotions are desirable, but they are used to alleviate the adverse impact of the lack lender commitment to future credit limits. Enhancing commitment via regulatory reforms could improve welfare because the equilibrium outcome does not implement the constrained optimal allocation with such a commitment.

To the best of our knowledge, our paper is the first one to document promotional pricing of credit card debt and the first one to examine its consistency with the predictions of the equilibrium theory of intertemporal pricing of unsecured credit lines. We see our work as the first step to explore further ramifications of this neglected feature of the data, which ultimately may improve the performance of the existing models of unsecured lending and uncover the kind of features of the environment that are key to understanding the data more broadly. In a complementary work,

[7]The optimality of borrower protection requires that the regulator takes a paternalistic approach and protects the consumers from her own self. The regulators have to take a stand that the consumer's ex-ante preferences ought to be maximized rather than her ex-post preferences.

Ausubel and Shui (2005) study result from an experiment of mailing offers to consumers. They show the data from that experiment is consistent with revealed preference for contracts featuring low introductory rates and/or fees (the exact terms are not disclosed). Relative to that paper, we document the pricing of outstanding debt at large and establish the underlying theory. Agarwal et al. (2015) present related evidence regarding the trade-off between interest rates and fees.

The rest of the paper is organized as follows. Section 2 discusses the data. Section 3 presents the baseline theory and states the main result. Section 4 generalizes the model and studies the robustness the main result. Section 5 discusses the potential resolutions of the puzzle. Finally, Section 6 concludes.

2 A look at promo activity on credit cards

We begin by discussing the key empirical regularities pertaining to promotional credit card lending in the U.S. We focus on the two years preceding the Covid-19 crisis to characterize the credit card market under "normal" or "steady-state" economic conditions.

2.1 Data sources and description

Our data comes from the supervisory collection by the Federal Reserve System for the purposes of the Dodd-Frank Act Stress Test (DFAST).[8] The data comprises a panel of all general purpose credit card accounts reported by bank holding companies subject to DFAST in 2018 and 2019—which, according to estimates, covers about 70 percent of all credit lines out there.[9] The variables include the typical information seen on credit card statements short of an itemized list of purchases. There is no information about the borrower aside from information pertinent to the account. The reported statistics are based on a representative sample of accounts drawn from the full dataset by the data provider and it is fixed for research purposes using this dataset. We define credit card debt as credit card balances carried over for at least one billing cycle, which corresponds to one month.

[8]The dataset is confidential but it is available for all researchers within the Federal Reserve System. Replication codes are available from the authors upon a request.

[9]Bureau of Consumer Financial Protection estimates that the Y14M dataset we are using here covers about 70 percent of all outstanding card balances in the US (see CFPB (2019), page 18). The remainder of the market are cards issued by banks with assets of less than $100 billion, or cards issued by non-banks, such as credit unions, as such institution do not fall under DFAST.

We calculate debt for each month t by taking the difference between the balances on an account during $t-1$ month and subtracting any payments made by the borrower during month t. Whenever we report an interest rate on an account it pertains to the APR rate posted on that account.[10] A promotional account is an account that is flagged as promotional by the lender, with the expiration of the promotion being inferred when that flag disappears from the account.

2.2 Stylized facts

We summarize our findings into five stylized facts.

Fact 1: *About a quarter of general purpose credit card debt has an introductory promotional status; in most cases involving zero annual percentage rate (APR) for an introductory period of well over a year (on average).*[11]

Table 1 reports promotional credit card debt as a share of total credit card debt for all general purpose accounts and for accounts with prime credit scores.[12] The promotional debt accounts for 22.5 percent of the total credit card debt, and for prime borrowers that ratio stands at 27 percent. About 80 percent of promo accounts involves zero APR.

The last rows of the same table report the average time to expiration of existing promotions, their average duration, and the average credit score among both promotional and nonpromotional accounts. The baseline statistics are debt-weighted, with unweighted figures reported in parentheses. The average time to promo expiration at a point in time is about 9 months, but the length of the promotional spell nears 20 months when weighted by debt and 16 months when it is not weighted.[13]

Fact 2: *Promotional accounts involve a sizable APR hike after the promotional period expires, aligning the average APR after promo expiration with the average APR on nonpromotional accounts.*

[10] The data we use is proprietory but our results can be replicated within the Federal Reserve System and the codes are available upon request after appropriate clearances are obtained by the requesting party.

[11] Annual percentage rate (APR) refers to the yearly interest generated by a sum that credit card borrowers pay. APR is similar to the finance charge but it additionally includes any fees or additional costs associated with the transaction and does not take compounding into account.

[12] Prime credit score is 670 or above.

[13] Duration here pertains to effective duration; that is, it measures how long the promotional flag stays on these accounts in a continuous fashion. This measure may involve an extension of the initial promotion and hence does not necessarily imply that this is the duration of the promotional offerings per se.

Table 1: Promotional debt: prevalence and duration.

Statistic[a] [in % unless otherwise noted]	2019	2018
Fraction of debt with promo rate[b]	22.3	22.4
Fraction of prime debt with promo rate[c]	27.3	27.0
Average time to promo expiration[d] [in months]	9.6 (8.3)	8.3 (7.5)
Average duration of promo spells[d] [in months]	19.8 (15.7)	20.4 (16.5)
Fraction of zero APR promos[a]	80.4	83.3
Fraction of promos with APR $\leq 3\%$	84.1	85.7
Fraction of promos with APR $\leq 6\%$	88.1	89.6
Average credit score on all promo accounts	727	728
Average credit score on zero APR promo accounts[e]	731	726
Average credit score on nonpromotional accounts	696	698

[a]We calculate each respective statistic for each month in 2018 and 2019 and then average them over each respective year. [b]We define debt as credit card balances that are carried over for at least one cycle. We calculate it on the account level in each month t by taking the difference between the balances in month $t-1$ net of payments made by the borrower in month t. [c]Prime debt includes accounts with prime credit score (e.g., minimum 670 credit scores on the account). [d]Debt-weighted, unweighted values are in the parentheses. [e]Most aggressively discounted promotional cards: 0 APR with 3 percent or less balance transfer fee Source: Federal Reserve System, Y14M.

The gap between the introductory promotional rate and the later reset rates on promotional accounts is about 16 percentage points on average (promo discount, hereafter). The median promo discount is similar to the mean, and even the 10th percentile discount is sizable, at about 5-6 percentage points. This shows that most promo accounts feature a large jump in the interest rate after the introductory period ends. The last rows of the same table shows that the average APR on nonpromotional accounts is similar to the average reset rate on promotional accounts.[14]

Fact 3: *Delinquency rate on promotional debt is about average as for the credit card market as a whole.*

Table 3 reports delinquency rates on promo credit cards for three different periods of such an account's life cycle: 2 months before, 2 and 5 months after the expiration of the promo period. Delinquency rate is calculated as the fraction of debt on these cards that is 30+ days past due and 120+ days past due and has not (yet) been written off by the lender (i.e., among accounts that remain on lenders' books). Delinquency rate is measured as a percentage of outstanding debt.[15] As we can

[14]Nonpromotional accounts are accounts that could have been promotional in the past. The sample includes non-promotional at the time of measurement.

[15]These statistics generally do not include accounts that are 180+ days past due and without any payment, nor accounts discharged via bankruptcy, since this leads to a statutory write off and the account vanishes from our dataset.

Table 2: Cost of promotional debt.

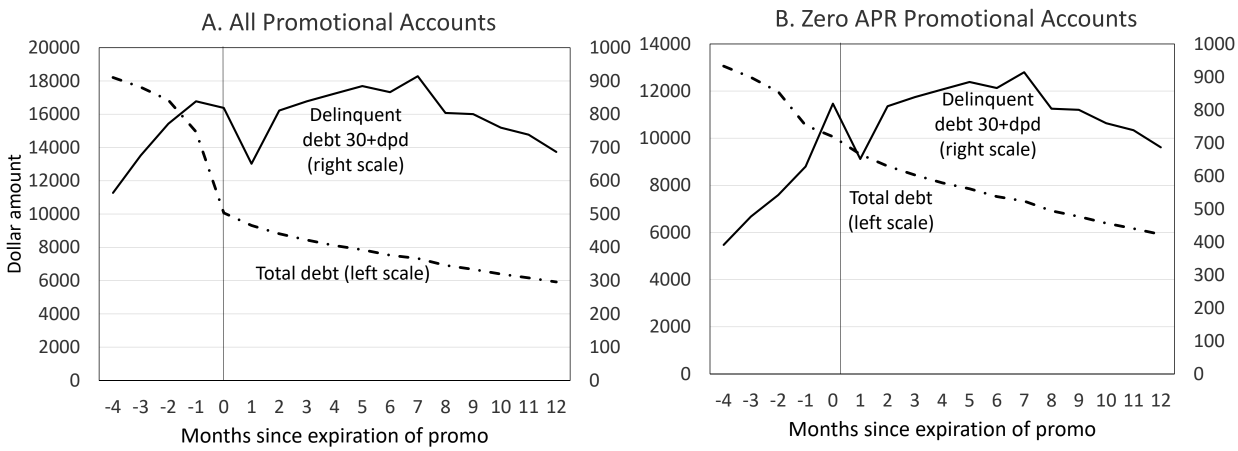
Statistic [in %]	2019	2018
Average APR discount vis-à-vis the later reset rate[a]	16.8	17.1
- 10th percentile of APR discounts on promo accounts	4.83	6.0
- 50th percentile of APR discounts on promo accounts	17.3	18.0
- 90th percentile of APR discounts on promo accounts	25.0	25.2
Average APR on nonpromotional accounts[c]	18.9	18.2
Median APR on nonpromotional accounts	18.1	17.6
Average APR on accounts that were never promotional[d]	18.7	18.0
Median APR on accounts that were never promotional	18.0	17.3

Notes to previous tables apply. [a]Debt-weighted statistics (reported percentiles are thus effectively for dollars of outstanding debt as a unit of observation). [b]APR discount is the difference between the promotional APR on the account and the nonpromotional reset rate on the same account. [c]Nonpromotional account as of the billing cycle at the time of measurement. [d]Nonpromotional since origination of the account as of the billing cycle at the time measurement. To obtain this statistic we average monthly statistics throughout our sample period (years 2018 and 2019).

see, comparing to nonpromotional accounts, delinquency on promotional accounts is lower before the expiration of the promo period but it is substantially higher afterwards.

Comparing the delinquency rate on promotional accounts to nonpromotional accounts poses some challenges. Since zero APR debt involves no payments at all, delinquency during the promotional period is less likely simply because low or no payments are required by the lenders. On the other hand, the calculation of the delinquency rate after the expiration of promotional period involves debt repayments due to a rate hike, with the repayment of debt (the denominator) among current accounts being a potential factor that drives up our delinquency rates. To address this issue, Figure 1 depicts the decomposition of the delinquency rate to its numerator (delinquent debt) and its denominator (total debt). The figure indicates that the delinquency rate on promotional debt is about average in comparison to nonpromotional accounts; debt repayments, while an important factor, are only partially driving up the rise in delinquency rates after the expiration of the promo period—especially in the case of zero APR accounts.

Figure 1: Delinquent debt and total debt at promo flag expiration.



Notes: The figure plots debt and delinquent debt that is 30+ days past due and has not (yet) been written off (typically after 180 days past due or after bankruptcy discharge). The left panel includes all promotional cards and the right panel reports the same for the most aggressively discounted promotional cards (0 APR cards with 3 percent or less balance transfer fee). The pool of accounts is fixed and they come from different time periods in 2018 and 2019, all centered around the expiration of the promo period ("0" on the horizontal axis). Source: Federal Reserve System, Y14M.

While it is possible that some delinquent debt is recovered later on, as long as recoveries on promotional accounts are no higher than on all accounts, the 3 percent balance transfer fee would not be sufficient to cover the default losses suffered by lenders on the most aggressively priced promotional accounts (those labeled as zero APR).[16] Our data thus points to the conclusion that lenders are on average losing money on the aggressively priced accounts during the promotional period, albeit they may and likely do break even in present value by charging the reset rates after the expiration of the promotional period. In fact, the evolution of debt on these accounts shown in the same figure shows that a large fraction of debt remains unpaid for months after the promo period ends.

Fact 4: *There is no systematic change in the default risk posed by borrowers between promo origination and its expiration.*

The average credit score on promo accounts is 732 in the first 3 months after the start of the promo period and 736 in the first three months after the end of the promo period.[17] Similarly, the account level difference in credit score between promo expiration (3 months average after expiration) and promo origination (3 months average after origination) is 3 points. The median is also

[16]The charge off rate on credit card debt reported by the Federal Reserve Board of Governors was above 3 percent during this time period.

[17]Since the credit scores are sensitive to monthly changes in the credit card utilization we look at their 3-month averages around the events rather than a 1-month average.

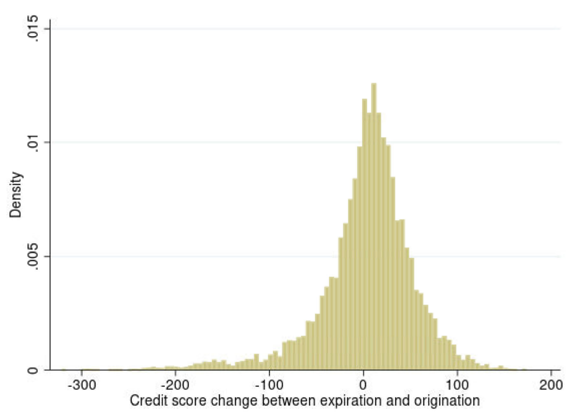
Table 3: Average delinquency rates on credit card debt.

Statistic [in %]	30+ dpd[a]	120+ dpd
All promo accounts:		
- 2 months before expiration of promo	4.6	2.7
- 2 months after expiration of promo	9.2	5.6
- 5 months after expiration of promo	11.3	7.0
0 APR promo account (3% or less BT fee):[b]		
- 2 months before expiration of promo	4.5	2.8
- 2 months before expiration of promo	9.2	5.6
- 5 months before expiration of promo	11.3	7.0
All accounts	6.7	3.5
Nonpromo accounts[c]	7.9	4.2

Notes to previous tables apply. [a]30 or more days past due credit card debt that has not been written off by the lender. Delinquent credit card debt is generally written off after 180 days past due and after debt is discharged in bankruptcy court.[b]This category includes most aggressively priced promo account; that is, those with zero APR and 3 percent or less balance transfer fee. [c]Accounts that are nonpromotional at measurement; together the two categories cover all accounts.

positive, at 8 points. This indicates that, if anything, the average or median borrowers' riskiness goes down during the promotional period, albeit insignificantly. Figure 2 plots the histogram of score changes in our data. (The score change is an unweighted statistic calculated across all promo accounts throughout the sample period.)

Figure 2: Histogram of credit score changes between promo origination and expiration.



Notes: The figure plots the histogram of the changes in credit scores across promo accounts during the promo period. To calculate it, we take the average score on the account over the first 3 months after the expiration of the promotion and subtract the average score on the same account over the first 3 months after the origination of the promotion. The score change is an unweighted statistic calculated across all promo accounts throughout the sample period. Source: Federal Reserve System, Y14M.

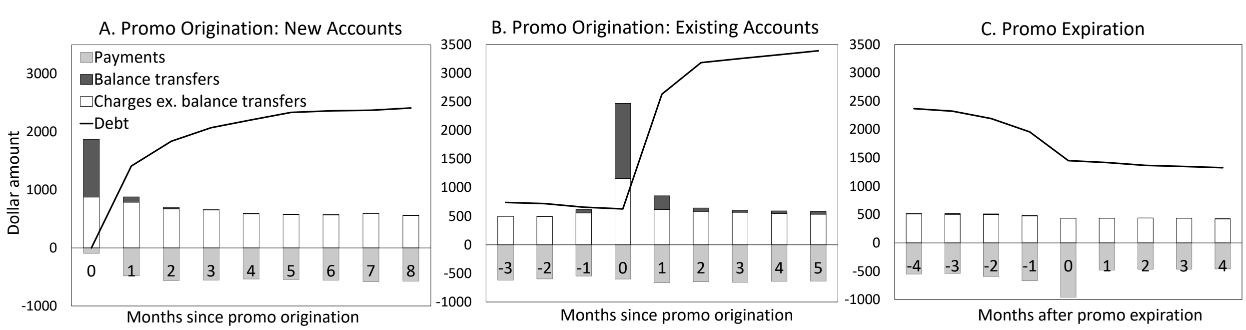
Fact 5: *Most debt on promo accounts originates from promotional balance transfers from other cards, with volume of inbound promotional balance transfers roughly matching the flow of expiring* *promotional debt.*[18]

The first two panels of figure 3 plot charges and balance payments on a cohort of accounts originated in early 2019 and tracked until early 2020, both for new originated promo accounts (panel A) and existing nonpromotional accounts that become newly promotional (panel B). The solid line plots the stock of accumulated debt on these accounts, which is the result of an accumulation of the depicted charges from the previous months net of total payments made until this particular month. As we can see, debt jumps immediately in the first month and then plateaus over the duration of the promo period. Balance transfers are the key driver of charges early on and also the accumulation of debt on these accounts. Expiration of the promo period is associated with accelerated debt repayments (panel C). That being said, a significant fraction of promo debt remains unpaid even after the promo period expires.

Overall, the annualized flow of promotional balance transfers is about 15 percent relative to total card debt outstanding. This is reported in Table 4. Assuming the average expiration of a typical account of about 18 months, the volume of balance transfers is roughly in line with the volume of expiring promotional debt.

While we do not observe outbound balance transfers in our data, the observed jump in balance payments around the expiration of the promo status is consistent with the idea that balance transfers are also accelerating debt repayments around the time promo period expires.

Figure 3: Charges and payments over the life cycle of promo cards.



Notes: The figure shows the life cycle of new promotional accounts and newly promotional existing accounts. We plot monthly charges excluding balance transfers, such as fees, purchases, cash advances (white bar), inbound balance transfers (black bar), and balance (re)payments (grey bar). Accumulated debt is the cummulation of charges, balance transfers and payments. Source: Federal Reserve System, Y14M.

A balance transfer typically carries an additional fee, even if it is promotional. This creates an additional source of revenue for lending even when the APR on the promo account is zero. Table 4

[18] A promotional balanced transfer features an introductory promotional rate.

reports the fees charge on balance transfers. As we can see, typical balance transfer involves a fee equal to 3 percent of the transferred amount, and in as many as 15 to 20 percent of cases balance transfers are for free. For comparison, the average charge-off rate on credit card accounts during the sample period was about 3.5 percent, which measures the fraction of debt deemed unrecoverable after 180 days overdue, net of the flow of recoveries.[19] We have already shown that delinquency rates on promo accounts are not significantly different than those seen on nonpromo accounts. This indicates that even with balance transfer fee lenders are likely losing money on the most discounted zero APR cards during the promotional period.

Table 4: Balance transfers to promotional cards.

Statistic [in %]	2019	2018
Annual balance transfers to total card debt[a]	15	14
Annual balance transfers to promo card debt[a]	69	64
Fraction of balance transfers that are promotional	94.2	94.4
- with zero balance transfer fee[b]	14.7	19.8
- with balance transfer fee $\leq 3\%$	65.0	68.6
- with balance transfer fee $\leq 6\%$	99.9	99.9

Notes to previous tables apply. [a]Balance transfers pertain to inbound balance transfers (flow of balances coming in from some other account). [b]As a fraction of all promotional balance transfers.

3 Theory of credit line pricing

This section lays out the baseline theory of credit line pricing and derives our main result. We relax some of the assumptions made here in Section 4 and disuss further extensions in Section 5.

3.1 Environment

Time is discrete and there are three periods, $t = 1, 2, 3$. The economy is populated by a large number of consumer families and lenders. A consumer family comprises a mass 1 of identical members (shoppers) who fully share all consumption risks. There is no aggregate uncertainty and

[19]Charge-offs in 2018 and 2019 oscillated between 3.5 and 4 percent on annual basis according to the data published by the Board of Governors of the Federal Reserve System (Charge-Off and Delinquency Rates on Loans and Leases at Commercial Banks). Charge-offs, which are the value of revolving loans removed from the books and charged against loss reserves, are measured net of recoveries as a percentage of average loans and annualized.

the law of large numbers is assumed. Information is symmetric and lenders perfectly observe consumer's state.

Each member of a family starts with income y and debt $b_0 > 0$, and with probability $0 < p < 1$ the income of all members of a given family switches to a low realization $y - \Delta$ in period 2, or period 3, where $0 < \Delta < y$ denotes the size of the negative income shock. The income shock of each family is independent and since it is perfectly coordinated across all members of the family it poses a noninsurable consumption risk. We refer to income level y as *high state* and income level $y - \Delta$ as *low state*.

The family evaluates consumption streams according to the expected utility function given by $\mathbb{E}\left\{u(c_1) + \beta u(c_2) + \beta^2 u(c_3)\right\}$, where, because of pooling of idiosyncratic consumption risks within the family, c_t pertains to the average consumption of its members in period t; $0 < \beta \leq 1$ is the discount factor, and u obeys the neoclassical assumptions.[20] We restrict attention to type-identical allocations on the family's member level, which means that decisions of each member are the same as long as its (exogenous) state is the same.

As a first pass, we assume that default on debt is nonstrategic and occurs in the low state; that is, the family intends to repay its debts, and it is the catastrophic nature of the negative income shock that leads to default. We later relax this assumption and generalize our results to allow for endogenous default. Endogenous default complicates the analysis but does not affect the results.

3.1.1 Credit contracts

Lenders compete in a Bertrand fashion and extend unsecured credit lines to individual family members to, in effect, maximize the utility of the consumer family subject to zero profits in expectation. A credit line is a vector $\mathcal{C} = (r, l, R, L)$; it comprises introductory terms specifying the introductory interest rate $r \geq 0$ and the introductory credit limit $l \geq 0$ (balance transfer offer), as well as the reset terms that also specify an interest rate $R \geq 0$ and a credit limit $L \geq 0$. The introductory terms apply in the first period after the contract is extended and they reset to R, L thereafter. There is no restriction on how r relates to R, or l relates to L, although we will be looking for promotional terms with $r \leq R$ to explain the data. Imposing $l \leq L$ will have no bearing on our results and so we do not need to make this assumption at this point. Each member of the family can hold one credit

[20] The flow utility u function is continuously differentiable, strictly increasing, and strictly concave and the Inada condition applies whenever consumption nears zero.

line at a time.

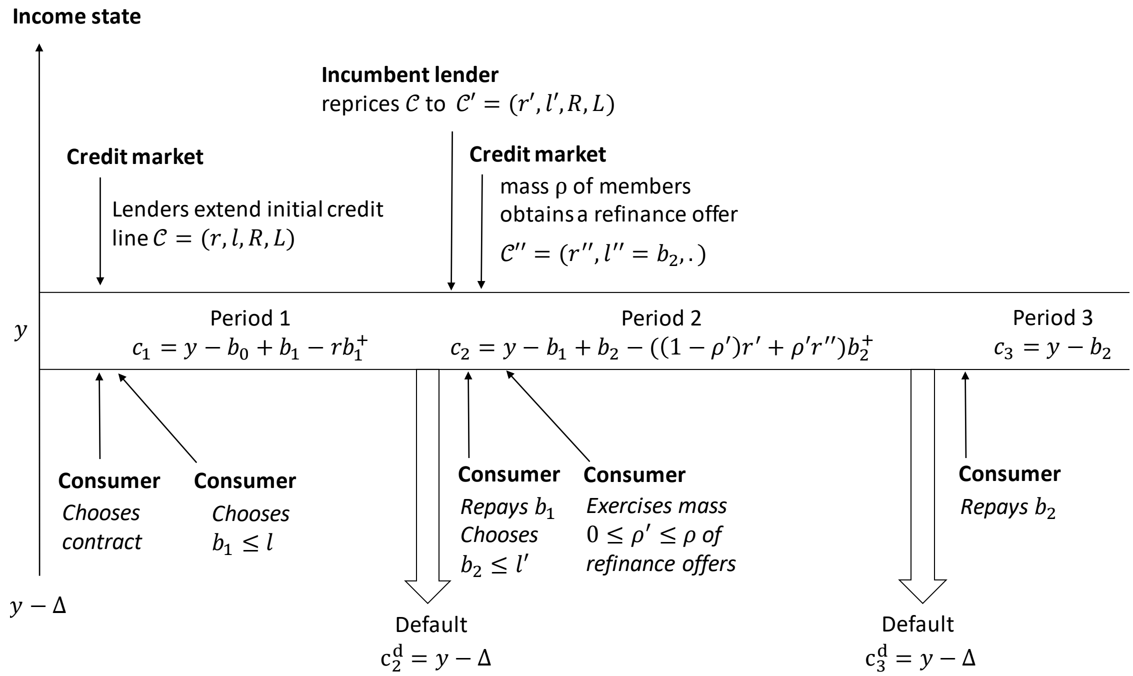
While lender are assumed to commit to these terms as of contract origination, they are allowed to "sweeten" terms later on; that is, we allow the lenders to set new promotions by reducing interest rate or raising credit limits at the onset of the second period. Credit limits can also be slashed when the line is not utilized, but the lender cannot force early debt repayment. These restrictions are consistent with Credit Card Accountability Responsibility and Disclosure (CARD) Act of 2009.[21] We refer to them as CARD Act restrictions and formalize them below.

3.1.2 Timing, decisions, and consumption constraints

The timing of events, decisions, and the resulting constraints for the consumer family are as follows (Figure 4):

[21] According to the Credit CARD Act of 2009 lenders must maintain interest rate on accounts up to five years. In particular, lenders cannot cut credit limits on the utilized portion of the credit line to force early debt repayment. They can also set promotional rates that expire after a specified period of time as long as it is longer than 6 months. Before the CARD Act, term changes were possible and did take place occasionally. However, anecdotal evidence still suggests that banks recognized the value of reputation and avoided changing terms. For example, Capital One and Citi, which together account for about thirty percent of the market, before the CARD Act of 2009 were contractually committing themselves to offer opt out options from any rate changes other than the ones triggered by noncompliance (e.g. late payments, overdraft etc.). In early 2008, Chase followed suit and also adopted an internal rule of not responding to any credit history changes unrelated to the account when reviewing terms. The OCC openly discouraged national banks from the practice of changing terms on credit cards (see OCC Advistory Letter, AL 2004-10). For this and other evidence, see Appendix to H.R. 5244 "The Credit Cardholders' Bill of Rights: Providing New Protections for Consumers," Hearing before the Subcommittee on Financial Institution and Consumer Credit of the Committee on Finance Services, U.S. House of Representatives, One Hundred Tenth Congress Session, April 17, 2008, Serial no. 110-109. Pages 280, 327, 371, 373-379, and 410 are of particular interest.

Figure 4: Timing, decisions, and consumption constraints.



Notes: The figure depics timing of events in the model and consumption constraints. The vertical axis pertains to the income state and the horizontal axis is time.

1. At the onset of the first period, members of the family shop for contracts and choose the initial credit line $\mathcal{C} = (r, l, R, L)$. There is an unimpeded access to credit in the first period. We restrict attention to type identical allocation and so all members obtain identical terms. After receiving the contracts, the consumer family chooses borrowing level $b_1 \leq l$ for each member and hence consumption of each member is

$$c_1 = y - b_0 + b_1 - rb_1^+, \tag{1}$$

where, throughout, we use notation $b_1^+ := \max\{b_1, 0\}$, and where rb_1^+ are interest payments made to the lender. If the income state switches to low in the second period, as assumed, the consumer family defaults on debt and each member's consumption is[22]

$$c_2^d = y - \Delta. \tag{2}$$

[22]Implicitely, we assume that initial lender retracts the unutilized portion of the credit line when default occurs and this way prevents the consumer from drawing more funds. The borrower cannot save borrowed funds from the first period and consume them in default state. This is a standard assumption but we later consider a generalization of the model that relaxes these assumptions and show that it has no bearing on our results.

2. If the high income state y persists to the second period, the incumbent lender reprices the initial credit line to $\mathcal{C}' = (r', l', R, L)$, which by the CARD Act restrictions discussed above requires $r' \leq R$ and $l' \geq b_1$. The consumer family then decides how much to consume in the second period, which amounts to borrowing repaying $b_2 - b_1$, and results in new level of debt given by $b_2 \leq l'$—and placed on the repriced line. After that decision is made, each member of the family applies to refinance debt b_2, which is not frictionless. Specifically, an individual member manages to obtain a refinance offer $\mathcal{C}'' = (r'', l'')$ with probability $0 < \rho < 1$ and $1 - \rho$ mass of members receives no offer. A refinance offer allows to transfer debt b_2 of an atomless member onto a new credit line, and without a loss a refinance offer features $l'' \geq b_2^+$ (this will turn out to be without a loss).[23] Consumption risk is shared among family members and so the second period consumption of each member is

$$c_2 = y - b_1 + b_2 - (1 - \rho')r'b_2^+ - \rho' r'' b_2^+, \tag{3}$$

where $0 \leq \rho' \leq \rho$ is the mass of refinance offers that the family chooses to exercise—in equilibrium we will have $\rho' = \rho$ or $\rho' = 0$ by linearity. If the income state switches to low in the third period, the consumer family defaults on debt and consumption is

$$c_3^d = y - \Delta. \tag{4}$$

3. If income y persists to the third period, the family repays, implying consumption in the third period under repayment is

$$c_3 = y - b_2. \tag{5}$$

3.2 Consumer problem

By backward induction, it is convenienient to state the consumer problem starting from the second period. It is assumed that the lender deals with many identical families and that the law of large numbers applies.

As of the second period, and starting in the high income state (repayment state), the consumer

[23] That is, the consumer family member request refinancing of b_2^+. As we show below, it is optimal for the second period lender to refinance in full.

family has accumulated debt b_1 (savings if negative) by that time and has a repriced incumbent's credit line $\mathcal{C}' = (r', l', R, L)$ on hand.[24] The family then chooses second period borrowing $b_2 \leq l'$ and exercises a fraction $0 \leq \rho' < \rho$ of mas ρ of refinance offers $\mathcal{C}''(\mathcal{C}', b_2) = (r''(\mathcal{C}', b_2), l'' = b_2)$ obtained by its members for the given b_2 to maximize

$$V(\mathcal{C}', b_1) = \max_{b_2 \leq l', 0 \leq \rho' \leq \rho} u(c_2) + \beta\left((1-p)\, u(c_3) + pu(c_3^d)\right), \tag{6}$$

where c_2, c_3, c_3^d are given by (3), (5) and (4), respectively. As of the first period, and given $\mathcal{C} = (r, l, R, L)$, the family chooses borrowing $b_1 \leq l$ to maximize

$$U(\mathcal{C}) := \max_{b_1 \leq l} u_1(c_1) + \beta\left((1-p)\, V\left(\mathcal{C}'(\mathcal{C}, b_1), b_1\right) + p\left(u_2\left(c_2^d\right) + \beta D\right)\right), \tag{7}$$

where c_1, c_2^d are given by (1) and (2), respectively, D is an exogenous continuation value after default severs the relationship with the existing lender, and where $\mathcal{C}'(\mathcal{C}, b_1)$ is the equilibrium repricing policy of the incumbent lenders.

3.3 Lender problem

Lenders are Bertrand competitors and extend contracts to maximize consumer's utility subject to zero profits. Lenders have deep pockets and, as a matter of normalization, they face zero cost of funding.

Denote consumer's policy function as $b_1(\mathcal{C})$, $b_2(\mathcal{C}'(\mathcal{C}, b_1), b_1) \equiv b_2(., b_1)$ and $\rho'(\mathcal{C}'(\mathcal{C}, b_1), \mathcal{C}''(\mathcal{C}'(\mathcal{C}, b_1), b_2), b_2) \equiv \rho'(., b_2)$, where $\mathcal{C}''(\mathcal{C}'(\mathcal{C}, b_1), b_2)$ is the second period lender's equilibrium refinance policy. The expected profit function of the first period lender is

$$\Pi(\mathcal{C}) := (r-p)\, b_1^+(\mathcal{C}) + (1-p)\left(1-\rho'(., b_2)\right)(r'-p)\, b_2^+(., b_1). \tag{8}$$

The lender's profit flow from the first period is determined by the excess interest over the default probability, $r - p$, which is paid on borrowing in the first period, b_1^+, and analogously for the second period.[25] The second period flow occurs only if the borrower repays, which takes place with

[24] This assumption is without loss, as we later show. See remark in footnote XX.

[25] The extensive form of the profit flow from the first period is as follows: $(r-p)\, b_1^+ = -b_1^+ + rb_1^+ + (1-p)\, b_1^+$, where "$-b_1^+$" is the loaned amount, "rb_1^+" are interest payments, and "$(1-p)\, b_1^+$" is the expected repayment. The

probability $1-p$, and applies to the fraction $1-\rho'$ of nonrefinanced credit lines. The second period lender's zero profit is

$$(r''-p)\, b_2^+(.,b_1) = 0 \Rightarrow r'' = p.$$

Consequently, the equilibrium interest rate on the refinanced credit line is p regardless of the amount and, as already mentioned, our earlier assumption that the second period lender does not impose a stricter credit limit than b_2 is without a loss.[26]

3.4 Equilibrium contracting problem

Bertrand competition implies that the initial lender's offer $\mathcal{C}$ maximizes family's expected utility subject to zero profits:

$$\max_{\mathcal{C}} U(\mathcal{C}) \;\; s.t. \;\; \Pi(\mathcal{C}) = 0. \tag{9}$$

We refer to a contract that solves the above as the *equilibrium contract* and characterize it. Before we do so, however, we restate the equilibrium contracting problem in a form that is amenable to analysis. To that end, we note the following key properties:

1) It is apparent from the setup that the ex post repricing of the initial contract can be built into the initial contract, since a rational consumer perfectly anticipates it and her decisions depend on the anticipated repriced terms rather than the original preset terms. Consequently, without a loss, we can assume that the the initial lender always offer the anticipated terms; that is, without a loss, we can assume $\mathcal{C}''(\mathcal{C}', b_2) = (R, L, .)$, but we must also restrict attention to the space of contracts that the lender would not find profitable to reprice ex post.[27] This will add a condition to the simplified

second period profit flow follows by analogy.

[26]Without this assumption the family could be force to choose to partially refinance the line, with a residual debt $b_2 - l''$ still remaining on the incumbent's partially refinanced lines. Since this would never occur in equilibrium, we assumed that a refinance offers are for an amount b_2^+ that borrowers request. The friction ρ is a technological constraint on the borrower's ability to access the market. An alternative approach would be to recast this friction as a search friction. However, this would require search costs being borne by lenders and priced into countracts. In such a case, the last condition would need to be modified to generate a strictly positive profit flow per accepted offer to cover the posting costs. It will later be clear that such an extention would make no difference in terms of results.

[27]The proof of this fact is straightforward. Suppose the lender offers $\mathcal{C} = (r, l, R, L)$ and the borrower expects this contract to be repriced ex post to $\mathcal{C}' = (r', l', R, L)$, where by CARD Act restrictions we must have $r' \leq R, l' \geq b_1$. By rational expectations, the borrowers expectations are correct and this is what happens ex post. The borrower will choose the exact same borrowing level b_1 and consumption c_1 if the initial contract is instead $\hat{\mathcal{C}} = (r, l, r', l')$, in which it is not repriced because the consumer's state is identical as in the first case. Accordingly, we have $U(\mathcal{C}) = U(\hat{\mathcal{C}})$ and $\Pi(\mathcal{C}) = \Pi(\hat{\mathcal{C}})$, which proves the claim because the restricted set of constracts that are not repriced ex post attains the same value of the program.

problem that we are about to write down.

2) It is clear from the linearity of the budget constraint in (3) that the refinancing decision has a bang-bang property, implying $\rho' = \rho$ if $R > p$, and $\rho' = 0$ otherwise.

3) We can eliminate the "kink" implied by the refinancing event in both the budget constraint and the intial lender's profit function. This is helpful not only to simplify the problem, but also to show that the contracting problem is globally differentiable and well-behaved (strictly concave subject to convex constraints). To that end, we define

$$R\left(\hat{R}\right) = \frac{1}{1-\rho} \max\left\{\hat{R} - p, 0\right\} + \min\left\{\hat{R}, p\right\}, \tag{10}$$

which, note, is a strictly increasing and non-negative for all $\hat{R} \geq 0$. It is clear that $R\left(\hat{R}\right) = \hat{R}$ for all $0 \leq \hat{R} \leq p$, and so the transformation has no bite unless $\hat{R} > p$. At $\hat{R} = p$ the function has a kink and steepens above that point. It is easy to verify that for the optimal ρ', which is $\rho' = \rho$ if $R > r'' = p$ and $\rho' = 0$ otherwise, the budget constraint in (1) after plugging in for R from (10) is $c_2 = y - b_1 + b_2 - \hat{R}b_2^+$ for all $\hat{R} \geq 0$. Similarly, the initial lender's zero profit becomes $\Pi = (r-p)\, b_1^+ + p\left(\hat{R} - p\right) b_2^+$ for all $\hat{R} \geq 0$. Furthermore, the function $R\left(\hat{R}\right)$ is a bijection, which allows to recover the original terms when needed.

The above properties allow us to recast the contracting problem as one involving a single lender who is extending a zero profit contract for the duration of all 3 periods with the terms given by $\hat{\mathcal{C}} = \left(r, l, \hat{R}, L\right)$. Since the consumer family shares refinance risk, such an equivalency should not surprise. What is relevant is how much the family pays to the lenders for credit. The same logic applies to the lenders. In particular, initial lenders can raise any targeted interest revenue by setting the reset rate appropriately thanks to the refinance friction $\rho > 0$. Put differetly, from their perspective, and thanks to that friction, whether refinancing occurs or not is irrelevant because both lenders operate under a zero profit condition and the whole problem can be consolidated into a single "stand-in" initial lender problem without a loss.

The lemma below formalizes this result by recasting (9) in an equivalent and simplified form. The proof of the lemma is immediate from (9) and we omit it. The transformed consumer problem is differentiable everywhere, the budget constraint is linear, and the objective function is strictly concave. Accordingly, constraints IC_1, IC_2, and CL of the maximization EQ in (11) list necessary and sufficient Karush-Kuhn-Tucker (KKT) conditions for the consumer problem, with the rest being a restatement of (9) after applying the above simplifications.[28] Importantly, the last restriction (condition 3 in the lemma) is needed to ensure that we do restrict attention to committed contracts that would not be repriced ex post. This condition must be verified for any candidate implementation satisfying EQ.

Lemma 1. *$\mathcal{C} = (r, l, R, L)$ is an equilibrium contract iff the following conditions are met:*

1) There exist $\hat{R} \geq 0$ such that $R = R\left(\hat{R}\right)$ and $\hat{\mathcal{C}} = \left(r, \hat{R}, l, L\right)$ solves

$$EQ: \max_{r,\hat{R},b_1,b_2,l,L} u\left(c_1\right) + \beta\left(1-p\right) u\left(c_2\right) + \beta^2\left(1-p\right)^2 u\left(c_3\right) + \beta U^d \tag{11}$$

where

$$BC \quad : \quad c_1 = y - b_1 + b_1 + r b_1^+, \; c_2 = y - b_1 + b_2 + \hat{R} b_2^+, \; c_3 = y - b_2, \tag{12}$$

subject to implementability constraints

$$\begin{aligned} IC_1 \quad &: \quad \left(u'\left(c_1\right)\left(1-r\right) - \beta\left(1-p\right) u'\left(c_2\right)\right) \mathbf{1}_{b_1 = l} \geq 0, \\ &\quad\;\; \left(u'\left(c_1\right)\left(1-r\right) - \beta\left(1-p\right) u'\left(c_2\right)\right) \mathbf{1}_{b_1 < l} = 0, \\ IC_2 \quad &: \quad \left(u'\left(c_2\right)\left(1-\hat{R}\right) - \beta\left(1-p\right) u'\left(c_3\right)\right) \mathbf{1}_{b_2 = L} \geq 0, \\ &\quad\;\; \left(u'\left(c_2\right)\left(1-\hat{R}\right) - \beta\left(1-p\right) u'\left(c_3\right)\right) \mathbf{1}_{b_2 < L} = 0 \\ ZP \quad &: \quad \left(r-p\right) b_1^+ + p\left(\hat{R}-p\right) b_2^+ = 0, \\ CL \quad &: \quad b_1^+ \leq l, \; b_2^+ \leq L, \end{aligned} \tag{13}$$

and where $\mathbf{1}_{b_1<l}$ is an indicator function that equals 1 if the subscripted constraint is true and 0 otherwise. ($U^d = p\left(u\left(c_2^d\right) + \beta D + \beta(1-p) u\left(c_3^d\right)\right)$ stacks constant terms associated with default state.)

3) The lender does not find it (strictly) profitable to reprice $r, l, \hat{R}, L$ under CARD Act restrictions.

Equilibrium interest rates r and $\hat{R}$ are determined only if there is borrowing. Our analysis

[28] If borrowing is on the constraint (e.g. $b_1 = l$), that constraint can be either binding or nonbinding, and hence the inequality in IC_1, IC_2. If $b_1 < l$ we require an interior solution, and hence the constraint then must hold with equality.

focuses on such a case, which turns out to be without a loss here. This is shown in the lemma below. The proof of this lemma is technical and it is in the appendix.

Lemma 2. *In equilibrium the consumer borrowers in both periods; that is,* $b_1 > 0$ *and* $b_2 > 0$.

3.5 Results

The proposition below states the main theoretical result of our paper: the equilibrium contract that satisfies Lemma 1 is $r = p = R(= \hat{R})$, with nonbinding credit limits and no refinancing taking place in equilibrium ($\rho' = 0$). The proof of the proposition is discussed in text below and only the more technical bits combined into the lemma below are relegated to the appendix.

Proposition 1. *Equilibrium consumption profile is constrained optimal and satisfies* $u'(c_1) = \beta u'(c_2) = \beta^2 u'(c_3)$. *The supporting equilibrium contract is* $r = p = \hat{R}(= R)$, l, L *nonbinding, and it uniquely implements the constrained optimal allocation among the contracts featuring* $r \leq R$ $\left(r \leq \hat{R}\right)$. *Furthermore, there is no refinancing in equilibrium; i.e.,* $\rho' = 0$.[29]

The key part of the proposition is the fact that the consumption profile should equalize the marginal utility across the periods up to discount β, that is, that $u'(c_1) = \beta u'(c_2) = \beta^2 u'(c_3)$ as stated. In light of the consumer's Euler equations (constraints IC_1, IC_2 in 13 under EQ in (11)), implementing this condition using a credit line with a nonbinding credit limit requires $r = p = \hat{R}$, where it is clear that such a contract satisfies the zero profit condition ZP in (13) for any b_1, b_2. It is also clear that this contract cannot be repriced to increase profits ex post, since $\hat{R}$ can only be lowered under the CARD Act restrictions and this can only lower profits for any $b_1 > 0$ (and any credit limit/borrowing). This implies that the requirements of Lemma 1 are satisfied as long as the associated consumption profile leads to the highest value of the program EQ in (11), which we discuss next.

Before we get to this, however, it is important to stress that among promotional contracts with $r \leq R$ this implementation is unique. As a result, promotional offers are unambigously suboptimal. The reason is as follows. While a binding credit limit l may deliver the condition $u_1' = \beta u_2'$ for $r < p$, which requires that borrowing is decreasing in r, the zero profit condition ZP in (13) then requires $\hat{R} > p$ (hence also $R > p$). But $\hat{R} > p$ i) either invalidates the second part of (**??**)

[29]By "inactive" we mean that the Lagrange multipliers assigned to IC_1, IC_2 constraints are all zero.

when L is nonbinding by implying $u_2' \neq \beta u_3'$ or, ii) if L is binding, it violates condition 3 of implementation under Lemma 1. To see these implications, note that part i trivially follows from IC_2 of the maximization EQ in (11) when L is nonbinding. Part ii applies when a binding L delivers both IC_2 and $u_2' = \beta u_3'$, but it violates condition 3 of the lemma because the lender will then find it (strictly) profitable to relax the credit limit L. Note that, by ZP in (13), the lender earns $\hat{R} - p > 0$ on the marginal dollar borrowed in the second period.

To prove the above proposition, we take an indirect approach of using an auxiliary planning problem featuring the same objective function but a slacker set of constraints than those in EQ—which applies because of Lemma 2 (the resource constraint involving saving in any of the periods is different). That planning problem also defines what we mean by *constrained optimum* in the proposition above.

In particular, let $c_1 = y - B$, $c_2 = y$, $c_3 = y$ be the autarkic consumption profile of the consumer, and define an auxiliary planning problem PL as follows:

$$PL : \max_{T_1, T_2, T_3} u\left(c_1 + T_1\right) + \beta\left(1 - p\right) u\left(c_2 + T_2\right) + \beta^2 \left(1 - p\right)^2 u\left(c_3 + T_3\right) + \beta U^d \tag{14}$$

subject to

$$RC : T_1 + \left(1 - p\right) T_2 + \left(1 - p\right)^2 T_3 = 0. \tag{15}$$

where, as in Lemma 1, we bunched together all the constants associated with default states into V^d. As we can see, the key property is that the planner maximizes the same objective function as the lender under Lemma's 1 EQ maximization but, unlike the lender, the planner can directly choose consumption in the high income state; that is, the planner does not need to respect constraints IC_1, IC_2 and CL.

What makes the above planning problem useful is how the underlying resource constraint RC relates to the equilibrium allocation. This is established in the lemma below, which is no more but a version of the Walras law. In particular, the lemma shows that, if the the consumer's budget constraint (BC) in (12) and the zero profit condition (ZP) in (13) are both satisfied, the planner's resource constraint (RC) in 15 is then satisfied for transfers that sustain the same level of consumption under PL. Furthermore, the converse is also true in the narrower sense: any zero profit contract that sustains the planner's consumption profile in the sense of satisfying constraints IC_1, IC_2 and

CL is a candidate solution under EQ because it also satisfies the consumer's budget constraint BC. The proof of the lemma relies on preservation of resources and it is in the appendix.

Lemma 3. *1) Suppose c_1, c_2, c_3 is the consumption profile that satisfies the constraints of the lender's maximization EQ in (11) and involves borrowing in both periods ($b_1 > 0, b_2 > 0$). Then, the implied transfers $T_1 = c_1 - (y - b_0)$, $T_2 = c_2 - y$, $T_3 = c_3 - y$ that sustain the same consumption profile under the maximization PL in (14) satisfy the planner's resource constraint RC in (15). 2) Conversely, consider c_1, c_2, c_3 and transfers $T_1 = c_1 - (y - b_0)$, $T_2 = c_2 - y$, $T_3 = c_3 - y$ that satisfy planner's resource constraint RC, and suppose there exists a contract $\left(r, l, \hat{R}, L\right)$ satisfying $l \geq -T_2 - T_3\left(1 - \hat{R}\right)$, $L \geq -T_3$, the constraints IC_1, IC_2 under EQ, and zero profit condition for $b_1 = -T_2 - T_3\left(1 - \hat{R}\right)$, $b_2 = -T_3$. Then, that contract sustains planner's consumption profile c_1, c_2, c_3 as a candidate solution under EQ.*

By the above lemma, if we find a unique solution to the planning problem PL in (14) and manage to identify a zero profit contract that i) the lender would not find profitable to reprice, and ii) which satisfies constraint IC_1, IC_2 and CL, we can be sure that we have identified the equilibrium contract. That contract is unique to the extent that there is no other contract that sustains the same consumption allocation. In particular, if there is another zero profit equilibrium contract that changes the allocation, that contract must satisfy RC and hence yield a strictly lower utility because the planner's problem, note, has a unique solution. However, nothing prevents the existence of multiple contracts that support the same allocation.

Consider now the necessary and sufficient conditions for the planning problem PL in (14).[30] Since this is a simple intertemporal consumption choice problem, the marginal conditions characterizing the solution involve the equalization of the marginal rate of substitution (MRS) implied by the objective function to the marginal rate of transformation (MRT) implied by the resource constraint RC in (15); in particular, it is easy to show that first order conditions involve:

$$MRS_1 := -\frac{u_1'}{\beta(1-p)u_2'} = -(1-p)^{-1} =: MRT_1, \tag{16}$$

$$MRS_2 := -\frac{u_2'}{\beta(1-p)u_3'} = -(1-p)^{-1} =: MRT_2, \tag{17}$$

[30] This is a standard concave programming problem featuring a linear constraints and a strictly concave objective function. There is a unique global maximum, and first order Lagrange conditions are both necessary and sufficient.

where u'_t denotes the marginal utility in period t. Consequently, the constrained efficient allocation satisfies $u'(c_1) = \beta u'(c_2) = \beta^2 u'(c_3)$, as stated in Proposition (1) for the equilibrium allocation. By Lemma (3), then, this proves the proposition because we have shown that the contract $r = p = \hat{R}(= R)$ with a nonbinding credit limit yields zero profits, satisfies constraints IC_1, IC_2 under EQ, and we have also noted that this contract will not be repriced ex post.

We have shown by now that the contracted interest rates should reflect the expected default risk in the period they apply to. In particular, the first period introductory rate should reflect default risk that applies to the first period. The equilibrium allocation is constrained optimal. In our baseline model, the default risk is fixed at p, and so this result implies a flat interest rate schedule $r = R$. Intuitively, the result comes about because, if a consumer draws an additional dollar of debt, she understands that she must pay for defaulting on that marginal dollar of debt. It is thus optimal for the consumer to choose a contract that ex post makes her internalize this cost ex post (after accepting the contract).

3.6 Relation to "tax smoothing theorem"

Basic economic theory tells us that a positive interest rate should lead to a deadweight loss that makes the allocation "distorted," and hence different from the allocation chosen by a planner who can directly control consumption. This is not the case, which highlights an important fact that distinguishes our result from the classic tax smoothing theorem (Atkinson and Stiglitz, 1976) in public finance. In particular, our results follow from the fact that positive interest corrects the distortion associated with a positive probability. The lack of distortion is implied by the fact that the problem is fundamentally different. In public finance the government needs to raise fixed tax revenue, while here it is all about reallocation of resources across periods under a zero profit condition that must make up for the risk of default.

The deadweight loss considerations do come into play away from the identified equilibrium contract. This is partially shown in the corollary below, which explicitely derives a condition that guarantees a zero profit contract underlying Lemma 1 (part 2). The corollary shows that in that case the set of transfers (planning solutions) that can be supported as equilibrium is restricted. We do not know whether the listed condition is an impossibility result at this level of generality, but our numerical experiments suggest this is the case.

Corollary 1. *For any candidate solution of PL in* (14) *satsfying RC in* (15)*, the associated consumption allocation satisfies* IC_1 *and* IC_2 *in maximization EQ in* (11) *either for contract* $r = p = \hat{R}$ *with nonbinding credit limits, in which case BC and ZP in EQ hold, or satisfying BC and ZP in EQ requires the following necessary and sufficient condition:*

$$\frac{r-p}{\hat{R}-p} + \frac{1-r}{1-p} = -\frac{T_1}{T_2}(1-r)\,. \tag{18}$$

4 Generalizations and extensions

This section generalize our results by considering endogenous default and (hidden) savings. These features are natural extensions of the baseline setup with credit lines and need to be examined to ensure robustness of our results. We focus attention on characterizing the equilibria featuring positive borrowing $b_1 > 0, b_2 > 0$, since Lamma 2 is specific to the baseline setup. Lemmas (1) and (3) apply, and we will use them below. The proof of Lemma (3) for the extended setup can be found in the Internet Appendix and the proof of Lemma (1) follows from analogous arguments.

4.1 Hidden savings

Our baseline model assumes that consumers cannot borrow and save the borrowed funds in the first period to consume them in the second period after they default. While default wipes out debt, the consumer's consumption in the default state is equal to income. In practice, default may not imply immediate liquidation of assets. Debt collection and bankruptcy proceedings are time consuming, and in borrowers may be able to divert borrowed funds or unutilized credit lines to enhance their consumption in the low income state under default. We now extend the model to allow for such a possibility.

Suppose the consumers can simultanously borrow and save in each period, and assume savings cannot be seized by lenders under default. That is, the borrower has access to *hidden savings technology* that hides savings from bankruptcy court or debt collectors and makes them available for consumption during or after defaulting. We assume that the use of this technology is constrained by a fraction of net consumed borrowing the first period, and consider both when this constraint binds and when it does not bind. Formally, given contract $\left(r, \hat{R}, l, L\right)$, the consumer in the modified setup chooses $\left(b_t, b_t^d, c_t\right)_{t=1,2}$ to maximize

$$U\left(c_1, c_2, ...\right) = u\left(c_1\right) + \beta\left(1-p\right)u\left(c_2\right) + \beta^2\left(1-p\right)^2 u\left(c_3\right) + \beta U^d\left(b_1^d, b_2^d\right) \tag{19}$$

subject to

$$\begin{aligned} c_1 &= y - b_0 + b_1 - r\left(b_1 + b_1^d\right), \\ c_2 &= y + b_{1d} - \left(b_1 + b_{1d}\right) + b_2 - \hat{R}\left(b_2 + b_2^d\right), \\ c_3 &= y + b_2^d - \left(b_2 + b_2^d\right), \\ b_1^+ &\leq l, b_2^+ \leq L \end{aligned} \tag{20}$$

and the hidden savings constraint given by

$$b_1^d \leq \tau b_1\left(1-r\right), b_2^d \leq \tau b_2\left(1-\hat{R}\right), \tag{21}$$

where the value from defaulting is endogenous and depends on b_1^d, b_2^d via

$$U_d\left(b_1^d, b_2^d\right) = p\left(u\left(y - \Delta + b_1^d\right) + \beta D + \beta\left(1-p\right)u\left(y - \Delta + b_2^d\right)\right). \tag{22}$$

In the consumer's budget constraint above, as of the first two periods, $b_1+b_1^d$ is the total borrowed amount in the first period, where b_{1d} are consumer's hidden savings for the future period that do not accrue to consumption in the first period; isntead, b_{1d} is consumed in the second period under default, as implied by (22). b_1^d also accrues to consumption in the second period's high state, but in that case the same amount is paid back to the lender, for a null net effect on consumption in that state (as implied by the term "$b_1^d - \left(b_1 + b_1^d\right) = -b_1$" in the equation for c_2). The hidden saving constraint in (21) states that only a fraction τ of the consumed borrowing net of intreest payments can be hidden this way. The budget constraint from the second period onward is analogous.

To characterize the constrained optimal allocation, we analogously use an auxilizary planning problem under which the planner chooses lump-sum transfers T_1, T_1^d, T_2, T_2^d and T_3 to maximize

$$\begin{aligned} U^{PL}\left(T_1, T_2, ...\right) = & u\left(c_1+T_1\right)+\left(1-p\right)\beta u\left(c_2+T_2\right)+\left(1-p\right)^2\beta^2 u\left(c_3+T_3\right) \\ & +\beta p\left(u\left(c_2^d+T_2^d\right)+\left(1-p\right)\beta u\left(c_3^d+T_3^d\right)\right) \end{aligned}$$

subject to

$$T_1+\left(1-p\right)T_2+pT_2^d+\left(1-p\right)\left(\left(1-p\right)T_3+pT_3^d\right)=0 \tag{23}$$

and we impose the following additional constraints on the planner to mimick the effect of the the hidden savings constraints above:

$$T_2^d \geq \tau T_1+\kappa_1, T_3^d \geq \tau T_2+\kappa_2, \tag{24}$$

where κ_1, κ_2 are constants that match the hidden saving constraint in terms of levels. As we will see, the marginal conditions involving the last constraint will not depend on constants κ_1, κ_2, and so introducing these constants will turn out to be without a loss. As before, $c_1 = y - b_0$, $c_2 = c_3 = y$, $c_2^d = c_3^d = y - \Delta$ is the autarkic consumption allocation.

The reason why the constraint in (24) captures the same trade off as the constraint in (21) is because, on the margin, the planner can increase insurance in the low income state by increasing consumption transfer in the first period. However, since T_1 does not correspond to net borrowing in that period, the constant κ_1 is needed to match the levels of the two constraints.

Binding hidden savings constraint.— We begin by analyzing the case when the hidden saving constraint binds. Using (23) under the assumption that it binds, we combine equations (23) and (23) to eliminate T_1 and plug in $T_2^d = -\tau\left(1-p\right)T_2 - \tau\left(1-p\right)^2 T_3 + \kappa$ where applicable. The relevant part of the lagrangian for the planning problem in the first two periods is

$$L = u\left(c_1+T_1\right)+\left(1-p\right)\beta u\left(c_2+T_2\right)+\beta p u\left(c_2^d-\tau\left(1-p\right)T_2\right)+...-\lambda\left(T_1+\left(1-p\right)T_2-...\right)$$

where λ is the lagrange multiplier on RC and it is the only constraint after plugging in the above. The first order conditions with respect to T_1, T_2 are

$$\begin{aligned} T_1 :& u_1' = \lambda \\ T_2 :& \beta (1-p) u_2' + \beta p \tau u_{2d}' = \lambda (1 - p (1 + \tau (1-p))) \end{aligned}$$

which us gives the analog of (16) for the extended model:

$$MRS_1 := \frac{u_1'}{\beta (1-p) u_2' + \beta p \tau u_{2d}' (1-p)} = (1 - p (1 + \tau (1-p)))^{-1} := MRT_1. \tag{25}$$

As for the consumer's Euler equation in the first period, we proceed by analogy and plug in $b_1^d = \tau b_1 (1-r)$. The relevant objective function for the choice of b_1 is

$$\begin{aligned} & u (y - b_0 + b_1 - r (b_1 + \tau b_1 (1-r))) + \\ & \beta (1-p) u \left(y - b_1 + b_2 - \hat{R} \left(b_2 + b_2^d\right)\right) + \beta p u (y - \Delta + \tau b_1 (1-r)) \dots \end{aligned}$$

which yields the Euler equation of the form

$$\frac{u_1'}{\beta (1-p) u_2' + \beta \tau p u_{2d}' (1-r)} = (1 - (1 + \tau (1-r)) r)^{-1}. \tag{26}$$

It is now clear from the comparison of the Euler equation in (26) and the planner's condition in (25) that the unique zero profit contract that supports the planner's allocation is $r = p$, and by the analogous reasoning applied to the marginal conditions pertaining to last two periods, that $\hat{R} = p$. Clearly, this contract satisfies the zero profit condition, which here is given by

$$(r - p) \left(b_1 + b_1^d\right) + (1-p) \left(\hat{R} - p\right) \left(b_2 + b_2^d\right) = 0.$$

Note that any other interest rate r would result in a different consumption level relative to that of the planner, implying a lower value of that program. As mentioned, an analog of Lemma 3 applies. (The proof of this fact is in the Internet Appendix.)

We next analyze what happens when the hidden savings constraint does not bind.

Nonbinding hidden savings constraint.— If (24) does not bind, the planner strives to equalize the marginal utility from consumption in the second period's high state and the second period low state. That is, the allocation features $u_2' = u_{2d}'$ (the consumer is fully insured in that sense). This is easy to see from the first order conditions implied by the corresponding Lagrangian with respect to T_2 and T_2^d. We omit the details.

Accordingly, with this additional implication, the planner's marginal conditions for the first two periods assuming high income state are no different that those in the baseline economy, with $MRT_1 = -(1-p)^{-1}$ and $MRS_1 = -u_1' / ((1-p)\beta u_2')$, and analogously for the second period. As for the consumer problem, the Euler equation implied by the marginal condition associated with b_1 is identical as in the baseline model. Therefore, the contract $r = p = \hat{R}$ with nonbinding credit limits l, L is again a candidate supporting zero profit contract. What needs to be verified, however, is that the consumer Euler equation that applies to hidden savings b_1^d also implies full insurance for that contract; that is, that the consumer under this contract will indeed choose b_1 so that $u_2' = u_{2d}'$. But this is easy to see by rewriting the first period budget constraint as $c_1 = y - b_0 + (1-r)\, b_1 - r b_1^d$,which for $r = p$ immediately implies that the consumer optimally chooses consumption so that $u_1' = \beta u_{2d}'$. Since the Euler equation on the high income path for this contract implies $u_1' = \beta u_2'$, the result follows because the two equations imply $u_2' = u_{2d}'$, which was to be shown. We conclude by summarizing the results in the proposition below.

Proposition 2. *With hidden savings, the equilibrium contract is constrained optimal and features $r = p = \hat{R}$ and nonbinding credit limits, implying $\rho' = 0$.*

4.2 Endogenous default and time-varying default risk

We next consider the possibility of endogenous default. We simplify the baseline setup by assuming that income is constant at y; that is, $\Delta = 0$. Instead, we consider a shock $s > 0$ to the utility cost of default that determines repayment. In particular, if U is the continuation value from repaying and U^d is the continuation value from defaulting, the borrower defaults iff $U^d + s \geq U$, where s_t is i.i.d. random variable and the reasoning applies to each perio. We denote by p_2 the probability of default after the first period and p_3 the probability of default after the second period. These probabilities, note, are now endogenous. While this is a simplified version of our model, the analysis below shows that there is no interaction between endogeneity of default and the key argument that eliminates promotions in equilibrium.

Consider the consumer problem underlying the analog of EQ in (11). As in the baseline setup, starting from the second period, the consumer solves

$$V(b_1) = \max_{b_2 \leq L} u\left(y - b_1 + b_2\left(1 - \hat{R}\right)\right) + \beta \mathbb{E} \max\left[u(y - b_2), u(y) + s_2\right], \tag{27}$$

where $s_2 \geq 0$ is an i.i.d. random variable distributed according to cdf F_2 over which the above expectation operator is taken. As of the first period, the consumer solves

$$U := \max_{b_1 \leq l} u_1(y - b_0 + b_1(1 - r)) + \beta \mathbb{E} \max\left[V(b_1), V^d + s_1\right], \tag{28}$$

where $s_1 \geq 0$ is another i.i.d. random variable distributed according to cdf F_1 over which the above expectation operator is taken. The probability of default is endogenous and it can be defined as follows. Let $\overline{s}_1$ be the cutoff value such that $V(b_1) = V^d + \overline{s}_1(b_1)$;then, the probability of default in the second period is $p_2(b_1) = Pr(s_1 \geq \overline{s}_1(b_1))$. Analogously, in the third period, we have cutoff $\overline{s}_2$ that satisfies $u(y - b_2) = u(y) + \overline{s}_2$ and probability of default is $p_3(b_2) = Pr(s_2 \geq \overline{s}_2(b_2))$. While in full generality not much can be said about how p_2 compares to p_3, in a stationary model with commonly distributed shocks s we will have $p_2 > p_3$ only if $b_1 > b_1$. We assume distribution of s is such that $p_2 = p_3$ when debt is equal, implying the aforementioned property by assumption.

The Euler equation now involves an additional effect: more debt implies that the agent defaults with a higher probability. Specifically, taking the derivative with respect to b_1 gives:

$$u_1'(1 - r) = (1 - p_2(b_1)) u_2' - \beta \frac{d\overline{s}_1(b_1)}{db_1}\left(V(b_1) - V^d - \overline{s}_1(b_1)\right),$$

where the right-hand side comes from the differentiation of the expectation operator with respect to b_1, since

$$\mathbb{E} \max\left[V(b_1), V^d + s_1\right] = \int_{-\infty}^{\overline{s}_1(b_1)} V(b_1)\, dF_1(s) + \int_{\overline{s}_1(b_1)}^{\infty} \left(V^d - \overline{s}_1(b_1)\right) dF_1(s).$$

By definition of the cutoff, then, the Euler equation is identical as in the baseline model and given by $u_1'(1 - r) = \beta(1 - p_2(b_1)) u_2'$. The second period Euler equation follows by analogy and it is also the same as in the baseline model but with endogenous default probabilities, that is $u_2'\left(1-\hat{R}\right) = \beta\left(1-p_2\right)\left(1-p_3\right)u_3'$. Concluding, the consumer problem yields two conditions:

$$\frac{u_1'}{\beta\left(1-p_2\left(b_1\right)\right)u_2'} = \left(1-r\right)^{-1}$$
$$\frac{u_2'}{\beta\left(1-p_2\right)\left(1-p_3\right)u_3'} = \left(1-\hat{R}\right)^{-1}.$$

By analogy to the consumer problem, the planner maximizes

$$U^{PL}\left(T_1,T_2,T_3\right) = \max_{b_1\le l} u_1\left(y-b_0+T_1\right)+\beta\mathbb{E}\max\left[V^{PL}\left(T_2,T_3\right),V^d+s_1\right]$$

where $V\left(T_2,T_3\right) = u\left(y+T_2\right)+\beta\mathbb{E}\max\left[u\left(y+T_3\right),u\left(y\right)+s_2\right]$. The resource constraint is as in the baseline economy but with endogenous default probabilities:

$$C\left(T_1,T_2,T_3\right) := T_1+\left(1-p_2\left(T_2,T_3\right)\right)T_2+\left(1-p_2\left(T_2,T_3\right)\right)\left(1-p_3\left(T_3\right)\right)T_3 = 0,$$

where probabilities are defined analogously except that they use values taken from the planner's problem. (We omit the definition.) The Lagrangian to the above planner problem is of the form $L = U^{PL}\left(T_1,T_2,T_3\right)-\lambda\left(C\left(T_1,T_2,T_3\right)\right)$. Using the reasoning above, we know that the marginal rate of substitution involving the objective function is identical as in the baseline model for both the first and the second period and given by the the expression on the left-hand side of (16) and (17). As for the marginal rate of transformation, we consider the first two periods, and need to apply implicit differentiation to constraint function C above, which gives

$$MRT_1 = -\frac{\partial C/\partial T_1}{\partial C/\partial T_2} = \left(1-p_2-\frac{\partial p_2}{\partial T_2}T_2-\left(1-p_3\right)\frac{\partial p_2}{\partial T_2}T_3\right)^{-1}$$

and for the last two periods, we have

$$MRT_2 = -\frac{\partial C/\partial T_2}{\partial C/\partial T_3} = \left(\left(1-p_2\right)\left(1-p_3\right)-\frac{\partial p_2}{\partial T_3}T_2-\left(1-p_3\right)\frac{\partial p_2}{\partial T_3}T_3-\left(1-p_2\right)\frac{\partial p_3}{\partial T_3}T_3\right)^{-1}.$$

Comparing the planner's conditions to the consumer conditions, we can see that when credit limits do not bind, implementing the constrained optimum requires $r^{PL} > p_2$ and $\hat{R}^{PL} > p_3$ which solve $MRT_2 = \left(1-\hat{r}^{PL}\right)^{-1}$ and $MRT_2 = \left(1-\hat{R}^{PL}\right)^{-1}$, respectively, for planner's consumption profile (transfers). But such a contract is impossible to implement and would not yield planner's consumption profile in equilibrium because it yields strictly positive profits, which are given by

$$ZP : (r - p_2)\, b_1 + (1 - p_2)\left(\hat{R} - p_3\right) b_2 = 0.$$

This implies that such a contract violates the implementability conditions of Lemma (1). Accordingly, if the implementation of the constrained optimum is possible, it must involve binding credit limits, which, note, relax the requirement that Euler equations hold with equality. We now impose the following assumption to be able to consider these implementations. We assume that borrowing is decreasing in interest rates globally, which is intuitive and in regular parametric cases it is satisfied.

Assumption 1. *Assume with slack credit limit policy functions of the consumer obey: i)* $db_1/dr \leq 0$*, i)* $db_2/d\hat{R} \leq 0$*, and iii)* $db_1/d\hat{R} \leq 0$*, where by "d" we mean total derivatives.*

With Assumption 1 in place, the following (limit) implementation is possible: 1) Set $r = p = \hat{R}$ and assume that at $\hat{R}$, by being indifferent, the lender will not ex post relax the credit limit L. 2) Set l, L at a binding level to deliver planner's consumption. Since the zero profit condition holds, by Lemma (3) we have found our implementation.[31] (The proof of Lemma (3) for the extended economy can found in the Internet Appendix and strategy of the proof is identical.) We summarize the final result in the proposition below, and note that it requires the assumption above.

Proposition 3. *With endogenous default, under Assumption 1, the equilibrium contract is constrained optimal and features* $r = p = \hat{R}$ *and binding credit limits, implying* $\rho' = 0$*.*

5 Time inconsistency as a resolution of the puzzle

We finish by discussing selected resolutions of the puzzle. These resolutions draw on the insights from the previous section.

[31] if a lender whose profits do not change would rather relax the credit limit, the results still applies approximately. This is easy to see by considering a limit argument and set $\hat{R}$ arbitrarily below p.

5.1 Behavioral stories

Following up on the evidence and the analysis in Ausubel and Shui (2005), promotions can be rationalized in equilibrium by models of time inconsistency in consumption decisions. These models change borrower preferences and invalidate the logic that the borrower ex ante wants a contract that aligns with her ex post preferences. This is obviously a problem because the core argument behind our result is that the consumer chooses ex ante a contract that make her see the right trade offs ex post. We model the lack of self-control using the hyperbolic discounting framework (Laibson et al., 2007). We also comment on an alternative formulation that does not involve preferences.

The only modification relative to our baseline model pertains to consumer preferences. In particular, as of the first period, consumers in the hyperbolic model evaluate consumption streams according to

$$U(c_1, c_2, c_3) = u(c_1) + \beta\eta\mathbb{E}\left[u(c_2) + \beta u(c_3)\right], \tag{29}$$

but, as of the second period, they evaluate them again according to

$$u(c_2) + \beta\eta\mathbb{E}\left[u(c_3)\right]. \tag{30}$$

where $\eta < 1$ is an additional discount factor applied to the continuation value in every period. In simple words, the key difference here is that the consumer's ex ante preferences envision a "future self" who is more patient than the consumer will be ex post.

There are two formulations of the hyperbolic discounting model in the literature. The simplest one assumes a naivete consumer who is unaware of the subsequent change in preferences, and hence boudedly rational. That is, the consumer believes her "future self" will be more patient, as in (29), but ex post the consumer always becomes impatient. In contrast, the sophisticated formulation of the same model assumes that the consumer is aware that her preferences will change ex post. We discuss both cases below.

5.1.1 Naivete hyperbolic discounting

As mentioned, in this case the consumer erroneously believes that her future self will pay down debt faster than it will actually be the case. Formally, the consumer in the second period solves

$$b_2^{\eta} = \max_{b_2} \left[u \left(c \left(b_2; \mathcal{C}, b_1 \right) \right) + \eta \beta u \left(c_3 \left(b_2 \right) \right) \right],$$

subject to (12), where $\eta < 1$, but the first period self erroneously believes her future self will make choices without the extra discount η (or equivalently with $\eta = 1$). Lenders are aware of this error and their profit reflects b_2^{η} as opposed to $b_2 \equiv b_2^1$, implying the zero profit condition is:

$$ZP : (r - p) \, b_1^+ + (1 - p) \left(\hat{R} - p \right) b_2^{\eta +} = 0.$$

For any b_1, which is a state variable as of the second period, it is easy to see that $b^{\eta} > b_2$. Therefore, as long as $\hat{R} > p$, the lender expects to make more "excess" profits from the second period relative to what borrowers believes they will pay ($(\hat{R} - p) b_2^+$).

The mechanism through this model can generate promotions is now clear. Since the consumer erroneously believes she will borrow less in the second period, lenders can exploit it by loading onto the reset rate $\hat{R}$ to raise revenue and lower the introductory rate r. This appears attractive to the borrower ex ante because the borrower is unaware of the change in preferences. The proposition below summarizes this result and the proof shows that there exists a deviation that increases the value of the program from the flat contract $r = p = \hat{R}$.

Proposition 4. *Positive credit equilibrium features a promotional rate $r < p$ when $\eta < 1$.*

5.1.2 Sophisticated hyperbolic discounting

The sophisticated formulation of the hyperbolic discounting model is fundamentally different because the borrower's expectations about the future are rational. As we show, equilibrium in this case is undetermined and may or may not involve a promotion.

Let us first consider contracts with nonbinding credit limit L in the second period. Observe that the Euler equation of the future self is $u_2' \left(1 - \hat{R} \right) = \beta \eta \left(1 - p \right)) u$, while ex ante the consumer would like her future self to make decisions according to $u_2' \left(1 - \hat{R} \right) = \beta \left(1 - p \right) u_3'$. If credit limit does not bind, the ex-ante consumer's self would like the allocation to be such that $u_2' = \beta u_3'$, just as in the baseline model. This desired reset rate that supports this as an Euler equation solves $1 - \hat{R} = \eta(1-p)$, which implies $\hat{R} = 1 - \eta(1-p) > p$ Plugging in to the transformation in (10), and given $1 - \eta(1-p) > p$, we obtain $R = \frac{1}{1-\rho}(1 - \eta(1-p) - \rho p)$.

Of course, the zero profit condition now necessitates $r < p$, and by Corollary (1), which applies here with no changes, r must be such that it satisfies equation (18). Using the planning solution as in the baseline model, we know that constrained optimal allocation requires $u_1' = \beta\eta u_2'$ holds in the first period, since there is an additional discount factor. Satisfying the Euler equation $u_1'(1-r) = \beta\eta(1-p)u_2'$ and (18) may not be possible, and thus having $r < p$ requires a binding credit limit l to constrain b_1 and achieve allocation consistent with $u_1' = \beta\eta u_2'$ (if it is possible).[32] Of course, this can only happen when borrowing is decreasing in r so that a binding credit limit can implement planner's consumption.

The above contract is a valid supporting contact but it is not the only possible implementation of the constrained optimal allocation. The alternative implementation is to set L binding so that the condition $u_2' = \beta u_3'$ holds for $r = p = \hat{R}$, since in that case the lender will not have an incentive to relax that credit limit L ex post. We summarize the above results in the proposition below. We omit the proof showing that $u_1' = \beta\eta u_2' = \beta^2 u_3'$, since the proof is analogous to the baseline model but with modified preferences.

Proposition 5. *The constrained optimal allocation satisfies $u_1' = \beta\eta u_2' = \beta^2 u_3'$. 1) If consumer's unconstrained policy function $b_1(.)$ is decreasing in r, constrained optimal allocation can be implemented by the contract: i) $\hat{R} = 1 - \eta(1-p) > p$ (*implying $R = \frac{1}{1-\rho}(1 - \eta(1-p) - \rho p)$*), ii) L nonbinding, iii) $r < p$ satisfying (18), and iv) l binding to ensure $u_1' = \beta\eta u_2'$. Alternatively, 2) If consumer's unconstrained policy function $b_2(.)$ is decreasing in $\hat{R}$, the constrained optimum can also be implemented by the contract i) $r = p = \hat{R}$, ii) nonbinding l, and iii) binding L to ensure $u_2' = \beta u_3'$.*

One way to obtain determinancy of equilibrium with promotions is to restrict market incompleteness so that the second implementation is no longer viable. Imposing a condition $l \leq L$ as a restriction on contracts may go a long way to achieve this.

[32]Note that this is only possible as long as borrowing b_1 is decreasing in r, which under sensible parameterizations will be the case.

5.2 Time-varying default risk

Perhaps the most straightforward potential resolution of the puzzle is time-varying default risk. Note that we have shown that optimal contract prices in default risk in full. If this default risk is expected to decline as of the origination of the contract, this can lead to promotions.

However, there are two problems with this story. The first problem is the data, which shows that the key measure of default risk, credit scores, do not move systematically between the origination of promotions and their expiration. Note that all promotional accounts should improve in expectation if this story was responsible for promotions, as pricing reflects expected default risk as of the contract's origination.

The second issue is implied by the theory, which generally implies that those who borrow have low income and high debt, and thus expect income to increase and debt to be gradually paid back. As a result, our theory implies that the opposite should be true for borrowers on average: default risk should be expected to decline between contract origination and promo expiration. Our extended model featuring endogenous default in the previous section illustrated this property.

We thus conclude that quantitatively it would be difficult to argue that the large volume of promotions seen in the data can be attributed to expected fluctuations in the default risk.

6 Conclusions

We have shown that introductory promotional lending is ubiquitous in the U.S. credit markets and puzzling from the perspective of the canonical theory of consumption with unsecured debt and default option. We highlighted the potential of the hyperbolic discounting in this regards and identified its key limitations. We have left out the analysis of features such as asymmetric information for future work. Asymmetric information could help address the puzzle via either screening or adverse selection mechanisms, but it is not clear to us what kind of information problem promotional offers help overcome.

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Appendix

Omitted proofs

Proof of Lemma 2

Preliminaries: Let us compactly represent the consumer problem by the maximization: $U\left(\mathcal{C}\right) = \max_{b_1,b_2} \tilde{U}(\mathcal{C}; b_1, b_2)$, where $\tilde{U}$ is the same as the objective function of the maximization EQ in 9 but after plugging in from BC in 12 for c_1, c_2, c_3. Without a loss, note that the domain of b_1, b_2 can be restricted to a compact and connected set (low b_1 violated nonnegativity, and credit limits are always assumed finite on $\mathcal{C}$). The objective function is stricty concave.[33] By the the basic results in convex programming (without differentiability), there exists a unique policy function $b_1(\mathcal{C}), b_2(\mathcal{C})$ that maximizes $\tilde{U}$. Accordingly, the consumer problem is a strictly concave programming problem despite the "kink" at $b_1 = 0$, $b_2 = 0$. Furthermore, directional partial derivatives can used to obtain the necessary and sufficient condition for the maximum at $b_1 = 0$, $b_2 = 0$. Recall that $\mathcal{C} = \left(r, l, \hat{R}, L\right)$ is an equilibrium contract if it maximizes $U(\mathcal{C})$ subject to ZP: $(r-p)\, b_1(\mathcal{C}) + (1-p)\left(\hat{R}-p\right) b_2(\mathcal{C}) = 0$. We now return to the proof of the lemma. Step 1. (Particular case) Consider first the contract $\mathcal{C}^* = \left(r, \hat{R}, l, L\right)$ with $r = p = \hat{R}$, l, L nonbinding. Note that it is a zero profit contract for any b_1, b_2 by ZP in 13. We will show that the consumer borrows in both periods, i.e. $b_1(\mathcal{C}^*) > 0$, $b_2(\mathcal{C}^*) > 0$. To prove this, note that the consumer's Euler equations—given by IC_1, IC_2 in 13—imply $u_1'(c_1) = \beta u_2'(c_2) = \beta^2 u_3'(c_3)$, and hence

$$c_1 > c_2 > c_3. \tag{31}$$

Note that this applies regardless of whether the consumer borrows or saves (as noted above, at $b_1 = 0$ or $b_2 = 0$ we evaluate directional derivatives and obtain the same condition). By contradiction, i) suppose $b_1 > 0, b_2 \leq 0$. By (12), we know $c_1 = y - b_0 + b_1 - b_1^+ p$, $c_2 = y - b_1 + b_2 - b_2^+ p$, $c_3 = y - b_2$, and it is clear that in this case $c_3 > c_2$, which contradicts (31). ii) Suppose $b_1 \leq 0, b_2 > 0$. Note that the listed equations in part i now imply $c_1 < c_2$, which again contradicts (31). Finally, iii) suppose $b_1 < 0, b_2 < 0$, and note that it also contradicts (31) because the listed equations imply $c_3 > c_1$. This proves the claim for $\mathcal{C}^*$. Step 2. (General case) Next, consider any equilibrium contract $\mathcal{C}^{**} = \left(r, \hat{R}, l, L\right) \neq \mathcal{C}^*$ that solves (9) (after transforming the contract space). By contradiction, suppose the consumer does not borrow in one of the periods. In particular, i) suppose the consumer chooses $b_1(\mathcal{C}^{**}) \leq 0$, $b_2(\mathcal{C}^{**}) > 0$. By the zero profit condition ZP, $\mathcal{C}^{**}$ must feature $\hat{R} = p$ (since the profit flow from the first period is zero). But, if so, $b_1(\mathcal{C}^{**}), b_2(\mathcal{C}^{**})$ is also a feasible choice

[33] Let $\mathscr{C}\left(\mathcal{C}; b_1^i, b_2^i\right)$ be the budget set defined by BC constraints in 12 and nonnegativity of consumption. Consider $\left(b_1^i, b_2^i\right) \in \mathscr{C}\left(\mathcal{C}; b_1^i, b_2^i\right)$, $i = 1, 2$. Let $(b_{1\theta}, b_{2\theta}) = \theta\left(b_1^i, b_2^i\right) + (1-\theta)\left(b_1^i, b_2^i\right)$, and define period t consumption function $c_t(b_1, b_2)$ by the left-hand side of BC constraints in 12. As for the first period, note that $c_1(b_{1\theta}, b_{2\theta}) - \left(\theta c_1\left(b_1^1, b_2^1\right) + (1-\theta)\, c\left(b_1^2, b_2^2\right)\right) = -r\left(\theta b_1^{\theta+}\right) + r\theta b_1^{1+} + (1-\theta)\, r b_1^{2+} \geq 0$ by $\theta \max\left[b_1^1, 0\right] + (1-\theta)\max\left[b_1^2, 0\right] - \max\left[\theta b_1^1 + (1-\theta)\, b_1^1, 0\right] \geq 0$. Note that an analogous argument implies to the second period. Since u is strictly concave, the result now follows.

under $\mathcal{C}^*$ and it results in the exact same level of consumption (budget constraint equations are identical in that case at that point). Yet, for $\mathcal{C}^*$, the consumer made a different choice, since we have shown in step 1 that $b_2(\mathcal{C}^*) > 0$. We have now obtained a contradiction because the above shows

$$U(\mathcal{C}^*) > \tilde{U}(\mathcal{C}^*, b_1(\mathcal{C}^{**}), b_2(\mathcal{C}^{**})) = \tilde{U}(\mathcal{C}^{**}, b_1(\mathcal{C}^{**}), b_2(\mathcal{C}^{**})) = U(\mathcal{C}^{**}),$$

where, from the left, the first part (>) follows from strict concavity of the consumer problem, the second part (=) follows by the fact that budget constraint in BC implies the same consumption at $b_1(\mathcal{C}^{**}), b_2(\mathcal{C}^{**})$ for $\mathcal{C}^*$ and $\mathcal{C}^{**}$, and the last part (=) follows by the hypothesis. Accordingly, $\mathcal{C}^{**}$ cannot be an equilibrium as hypothesized because it does not maximize consumer's utility. Cases ii) and iii) follow by analogy. Q.E.D.

Proof of Lemma 3:

1) "⇒" Consider the given consumption profile c_1, c_2, c_3,and define the associated transfers under PL in (14) as follows $T_1 = c_1 - y - b_0$, $T_2 = c_2 - y$, $T_3 = c_3 - y$. By the hypothesis, c_1, c_2, c_3 satisfy BC in (12) and ZP in (13). Expanding the zero profit condition (the second line below) and plugging in from the budget constraint in (12) (the third line below) yields RC in (15) as claimed:

$$\begin{aligned}
\Pi = (r-p)\,b_1 + (1-p)\left(\hat{R}-p\right)b_2 &= 0 \\
(-b_1 + rb_1) + (1-p)\left(b_1 - b_2 + \hat{R}b_2 + (1-p)\,b_2\right) &= 0 \\
-(c_1 - (y - b_0)) - (1-p)(c_2 - y) - (1-p)^2(c_3 - y) &= 0 \\
\Rightarrow T_1 + (1-p)\,T_2 + (1-p)^2 T_3 &= 0.
\end{aligned}$$

2) "⇐" Let c_1, c_2, c_3 be the consumption profile that solves PL in (14) (that is, $c_1 = y - b_0 + T_1,\ c_2 = y + T_2, c_3 = y + T_3$, and T_1, T_2, T_3 solve PL). Given the supporting rate schedule $\left(r, \hat{R}\right)$ from the statement of the lemma, we use the budget constraint BC in (12) to back out the unique

values of b_1 and b_2 that ensure BC holds in periods 2 and 3 as follows:

$$\underbrace{y + T_3}_{c_3} = y - b_2 \Rightarrow b_2 = -T_3 \tag{32}$$
$$\underbrace{y + T_2}_{c_2} = y - b_1 + \underbrace{b_2}_{b_2=-T_3} \left(1 - \hat{R}\right) \Rightarrow b_1 = -T_2 - T_3 \left(1 - \hat{R}\right)$$

Next, we show that the first period budget constraint

$$c_1 = y - b_0 + b_1 (1 - r) \tag{33}$$

is satisfied for $c_1 = y - b_1 + T_1$ iff ($\Leftrightarrow$) the zero profit condition ZP in (13) holds for $b_1 = -T_2 - T_3 \left(1 - \hat{R}\right)$ and $b_2 = -T_3$, which, note, we have shown above satisfy (32). Plugging planner's consumption $c_1 = y - b_0 + T_1$ into (33), we obtain

$$y - b_0 + T_1 = y - b_0 + b_1 (1 - r).$$

Using the fact that $T_1 = -T_2 (1 - p) - T_3 (1 - p)^2$ by RC in (15), the above gives

$$\left(-T_2 (1 - p) - T_3 (1 - p)^2\right) = b_1 (1 - r).$$

Plugging in for T_2, T_3 from equations for b_2, b_3 in (32), we obtain

$$-\left(\underbrace{-b_1 + b_2 \left(1 - \hat{R}\right)}_{-T_2}\right) (1 - p) - \left(\underbrace{-b_2}_{T_3}\right) (1 - p)^2 = b_1 (1 - r),$$

which after basic manipulations yields the zero profit condition $b_1 (r - p) + \left(\hat{R} - p\right) b_2 (1 - p) = 0$, which holds by the hypothesis. This finishes the proof because the last expression is ZP in (13) and the reasoning works in reverse "$\Leftarrow$" (and hence a stronger "iff" statement applies as stated).

Proof of Corollary 1:

The proof is a modified version of the proof of the previous lemma (part 2). Take any feasible planning solution under RC with consumption profile c_1, c_2, c_3 and the implied transfers T_1, T_2, T_3, which implies 1) $c_1 = y - b_0 + T_1$, 2) $c_2 = y + T_2$, 3) $c_3 = y + T_3$ and 4) $T_1 + (1-p)\, T_2 + (1-p)^2\, T_3 = 0$. Analogously to the proof of the lemma, we use BC in (12), to back out b_1, b_2 as follows to back out b_1 and b_2 to ensure BC holds in period 2 and 3 for the planner's consumption profile, which gives

$$\underbrace{y - b_0 + T_1}_{c_1} = y - b_0 + b_1 (1-r) \Rightarrow b_1 = \frac{T_1}{1-r} \tag{34}$$

$$\underbrace{y + T_2}_{c_2} = y - \underbrace{\left(\frac{T_1}{1-r}\right)}_{b_1} + b_2 \left(1 - \hat{R}\right) \Rightarrow b_2 = \frac{T_1}{(1-r)\left(1-\hat{R}\right)} + \frac{T_2}{1-\hat{R}}$$

$$\underbrace{y + T_3}_{c_3} = y - \underbrace{\left(\frac{T_1}{(1-r)\left(1-\hat{R}\right)} + \frac{T_2}{1-\hat{R}}\right)}_{b_2} \Rightarrow \frac{T_1}{(1-r)\left(1-\hat{R}\right)} + \frac{T_2}{1-\hat{R}} + T_3 = 0$$

We now can be sure that if the last equation holds, and $r < 1$, $\hat{R} < 1$—which is inoccuous because otherwise we cannot have $b_1 > 0, b_2 > 0$—the consumer's budget constraint BC in (12) is satisfies. We next multiply both sides of RC in (15) by $(1-p)^2$ and subtract the resulting equantion side-by-side from $\frac{T_1}{(1-r)\left(1-\hat{R}\right)} + \frac{T_2}{1-\hat{R}} + T_3 = 0$ above, which eliminates T_3 and implies

$$b_2 = \frac{T_1}{(1-r)\left(1-\hat{R}\right)} + \frac{T_2}{1-\hat{R}} = \frac{T_1}{(1-p)^2} + \frac{T_2}{1-p}. \tag{35}$$

Next, we note that satisfying the zero profit condition ZP in (13) for b_1, b_2 given by (34) and (35) above requires

$$(r-p) \underbrace{\frac{T_1}{1-r}}_{b_1} + (1-p)\left(\hat{R} - p\right) \underbrace{\left(\frac{T_1}{(1-p)^2} + \frac{T_2}{1-p}\right)}_{b_2} = 0,$$

which after straightforward manipulations yields (18).

Proof of Proposition 4

Assume, by the way of contradiction, that the optimal contract is $r = p = \hat{R}$ $(R = p)$ with credit limits being slack or binding. We will show that, if $\eta < 1$, in either case, the lender has an incentive to deviate from this contract by lowering r and raising $\hat{R}$.

Consider now two economies: the first economy, referred to as the *hyperbolic economy*, is the economy with an additional discount $\eta < 1$ applied to the continuation value from the next period onward. As explained in text, preferences are time inconsistent because the the discount factor in the second period as of the first period is β, and becomes $\eta\beta$ only after the first period ends. The second economy, referred to as the *baseline economy*, is an economy with exactly the same discounts as of the first period, with the only difference being that these discounts do not change after the first period ends; that is, there is no time inconsistency and the ex ante consumer problem is exactly identical. In the hyperbolic economy the profit function is

$$\Pi^{\eta} := (r - p)\, b_1\left(r, \hat{R}, l, L\right) + (1 - p)\left(\hat{R} - p\right) b_2^{\eta}\left(r, \hat{R}, l, L\right),$$

where $b_2^{\eta}\left(r, \hat{R}, l, L\right)$ denotes the borrower's ex post policy function. In contrast, in the baseline economy, it is

$$\Pi := (r - p)\, b_1\left(r, \hat{R}, l, L\right) + (1 - p)\left(\hat{R} - p\right) b_2\left(r, \hat{R}, l, L\right),$$

where $b_2\left(r, \hat{R}, l, L\right)$ is the borrower's ex ante policy function. It is clear that the proof of Proposition 1 applies to baseline economy. This can be verified by repeating each step. Consequently, the equilibrium contract is $r = p = \hat{R}$, l, L nonbinding.

Consider now a deviation from $\hat{R}$ by some $d\hat{R} > 0$, applied to both economies and evaluated at that contract. By the implicit function theorem, the required offsetting change in dr^{η} to keep profits constant in the hyperbolic economy is

$$dr^{\eta} := -\frac{(1 - p)b_2^{\eta}(p, p, .)}{b_1(p, p, .)} d\hat{R}.$$

This can be calculated by implicitly differentiating the above profit function at $r = p = \hat{R}$. Assume the borrowing constraint in the first period is maintained slack if it was nonbinding initially and continues to be binding if it was binding initially ($\hat{R}$ is small enough not to affect the binding pattern of the constraint). We repeat the same calculation for the baseline economy, which removes superscript η from the above expression. To simplify, from now on we use shorthand notation and write b_1 instead of $b_1(p, p, .)$ and so on and so forth. We note the following

$$dr^\eta = \frac{(1-p)b_2^\eta}{b_1} d\hat{R} < \frac{(1-p)b_2}{\bar{b}_1} d\hat{R} = dr, \tag{36}$$

since, trivially, $b_2^\eta > b_2$, and $b_1 = b_1$, where dr is the required adjustment for the baseline economy. We also know that $U^\eta \equiv U$, since the ex ante consumer problem is identical across the two economies. The first order change in the consumer's utility implied by this deviation in the two economies can thus be written as follows:

$$\begin{aligned} dU^\eta &= \frac{\partial U^\eta}{\partial r} dr^\eta + \frac{\partial U^\eta}{\partial \hat{R}} d\hat{R} \\ dU &= \frac{\partial U}{\partial r} dr + \frac{\partial U}{\partial \hat{R}} d\hat{R}. \end{aligned}$$

Taking the difference side-by-side, and using the fact that $\frac{\partial U^\eta}{\partial r} = \frac{\partial U}{\partial r}$ (ex ante preferences are identical, $U^\eta \equiv U$), we obtain

$$dU^\eta - dU = \frac{\partial U^\eta}{\partial r} (dr^\eta - dr) > 0,$$

since we have shown in (36) that $dr^\eta < dr$ and we know $\frac{\partial U^\eta}{\partial r} < 0$ (r strictly contracts the budget constraint and the utility function is strictly increasing in consumption). We have now shown that there exists a deviation from the proposed contract that is profit feasible and raises the consumer's ex ante utility, a contradiction. We have also established the direction of this variation (lower r and higher $\hat{R}$).

Internet Appendix (not intended for publication)

Proof of Lemma 3 for extended setup from Section 4

Here we combine both extensions of the baseline model from Section 4 and prove the analog of Lemma 3.

1) "⇒" Consider the given consumption profile $c_1, c_2, c_3, c_2^d, c_3^d$,and define the associated transfers under PL as follows $T_1 = c_1 - y - b_0$, $T_2 = c_2 - y$, $T_3 = c_3 - y$, $T_2^d = c_2^d - y + \Delta$, $T_3^d = c_3^d - y + \Delta$. By the hypothesis, $c_1, c_2, ...$ satisfy the budget constraint and the zero profit constraint in . Expanding the zero profit condition (the second line below) and plugging in from the budget constraint in (12) (the third line below) yields RC in (15) as claimed:

$$\begin{aligned}\Pi = (r - p_2)\left(b_1 + b_1^d\right) + (1 - p_2)\left(\hat{R} - p_3\right)\left(b_2 + b_2^d\right) = 0\\ \left(-\left(b_1 + b_1^d\right) + r\left(b_1 + b_1^d\right)\right) + (1 - p_2)\left(\left(b_1 + b_1^d\right) - \left(b_2 + b_2^d\right) + \hat{R}\left(b_2 + b_2^d\right) + (1 - p_3)\left(b_2 + b_2^d\right)\right) = 0\\ -\left(c_1 - (y - b_0) + \left(c_2^d - y + \Delta\right)\right) - (1 - p_2)\left(c_2 - y + c_2^d - y + \Delta\right) - (1 - p_3)\left(c_3 - y + c_3^d - y + \Delta\right) = 0\\ -\left(c_1 - (y - b_0) + p_2\left(c_2^d - y + \Delta\right)\right) - (1 - p_2)\left((c_2 - y) - (1 - p_3)\left(c_3 - y + c_3^d - y + \Delta\right)\right) = 0\\ \Rightarrow T_1 + pT_2^d + (1 - p_2)\left(T_2 + (1 - p_3)T + p_3T_3^d\right) = 0.\end{aligned}$$

2) "⇐" Let $c_1, c_2, c_3, c_{2d}, c_{3d}$ be the consumption profile that solves the planning problem. Given the supporting rate schedule $\left(r, \hat{R}\right)$ from the statement of the lemma, we use the budget constraint BC in (12) to back out the unique values of b_1 and b_2 that ensure BC holds in periods 2 and 3:

$$\underbrace{y - \Delta + T_2^d}_{c_2^d} = y - \Delta + b_{1d} \Rightarrow b_1^d = T_2^d$$

$$\underbrace{y - \Delta + T_3^d}_{c_3^d} = y - \Delta + b_2^d \Rightarrow b_2^d = T_3^d$$

$$\underbrace{y + T_3}_{c_3} = y - b_2 \Rightarrow b_2 = -T_3 \tag{37}$$

$$\underbrace{y + T_2}_{c_2} = y - b_1 + \underbrace{b_2}_{-T_3}\left(1 - \hat{R}\right) - \underbrace{b_2^d}_{T_3^d}\hat{R} \Rightarrow b_1 = -T_2 - T_3\left(1 - \hat{R}\right) - T_3^d\hat{R}$$

Next, we show that the first period budget constraint

$$c_1 = y - b_0 + b_1 - r\left(b_1 + b_1^d\right) \tag{38}$$

is satisfied for $c_1 = y - b_1 + T_1$ and $c_2^d = y - \Delta + T_2^d$ iff ($\Leftrightarrow$) the zero profit condition ZP in (13) holds for $b_1 = -T_2 - T_3\left(1 - \hat{R}\right) - T_3^d\hat{R}$ and $b_2 = -T_3$, $b_1^d = T_2^d$, $b_2^d = T_3^d$, as defined above. Plugging in the planner's consumption $c_1 = y - b_0 + T_1$ to (38), and using the fact that $T_1 = -T_2^d p_2 - (1 - p_2)\left(T_2 + T_3^d p_3 + (1 - p_3) T_3\right)$ by RC, as well as the formulas for b_1, b_2, we obtain

$$\begin{aligned}
c_1 =& y - b_0 + b_1 - r\left(b_1 + b_1^d\right)\\
y - b_0 + T_1 = y - b_0 +& b_1 - r\left(b_1 + b_1^d\right)\\
\left(-T_2^d p_2 - (1 - p_2)\left(T_2 + T_3^d p_3 + (1 - p_3) T_3\right)\right) =& b_1 - r\left(b_1 + b_1^d\right)\\
-b_2^d p_2 - (1 - p_2)\left(-b_1 + b_2\left(1 - \hat{R}\right) - b_3^d\hat{R} + b_3^d p_3 - (1 - p_3) b_2\right) =& b_1 - r\left(b_1 + b_1^d\right)\\
(r - p)\left(b_1 + b_1^d\right) - (1 - p_2)\left(b_2\left(1 - \hat{R}\right) - b_3^d\hat{R} + b_3^d p_3 - (1 - p_3) b_2\right) =& 0\\
(r - p)\left(b_1 + b_1^d\right) + (1 - p_2)\left(\hat{R} - p_3\right)\left(b_2 + b_3^d\right) =& 0
\end{aligned}$$

which finishes the proof because the last expression is ZP andd the reasoning works in reverse "$\Leftarrow$" (and hence a stronger "iff" statement applies as stated).