Understanding Growth Through Automation: The Neoclassical Perspective*

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Abstract

We study how advancements in automation technology affect the division of income between capital and labor in the long-run. Our analysis focuses on the fundamental trade-off between the labor-displacing effect of automation and its positive productivity effect in an elementary task-based theory featuring a schedule of automation prices across tasks linked to the state of technology. We characterize the conditions necessary for the automation technology and technical change driving automation to be associated with a constant or increasing labor share. We show that balanced growth in the presence of automation is possible and identify the underlying task technology. We show that this technology is unique and aggregates to the Cobb–Douglas production function—thus providing novel task-based microfoundations for this workhorse functional form. We employ our framework to explore the potential linkages between the recent declines in the labor share and the nature of the current IT-powered wave of automation.

Key words: Automation, labor share, Uzawa's theorem, Cobb-Douglas production function, capital-augmenting technological progress, balanced growth

JEL Classifications: D33, E25, O33, J23, J24, E24, O4.

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The countervailing effects of automation on labor income have long been a cause of concern to policy-makers, economists and workers. On the one hand, automation displaces labor from tasks or activities, reducing labor income, but on the other hand, it is associated with productivity gains that reduce prices and increase the purchasing power of labor income. This mechanism is at the heart of automation's effect on labor's share in aggregate income, which has recently been in the spotlight due to its steady decline in the data.[1] In fact, one hypothesis holds that it is modern incarnation of IT-powered automation that has eaten into the labor share, raising the follow-up question of what might differentiate this wave of automation from past waves that did not seem to have the same effect on the labor share.[2]

Economic theory thus far offers only limited insights into how this elementary trade-off works, leaving a number of key questions without general answers. For example, is the negative effect of displacement always accompanied by a positive productivity effect? If so, what are the determinants of the latter effect in terms of technological fundamentals? Under what conditions, if any, is the net effect on the labor share positive? Can balanced growth with automation and constant factor shares be sustained and thus account for past growth experiences (i.e., the Kaldor facts)? Finally, is the current IT-powered wave of automation different in ways that would suggest a suppressed productivity effect?

Neoclassical growth theory hints at a possible answer but it is not helpful because it lumps all forms of capital deepening under the umbrella of aggregate capital and reduced-form aggregate production function. The existing microfoundations for the aggregate production function, such as those by Jones (2005) and Houthakker (1955), while fleshing out the microfoundations for the aggregate production function, do no readily provide an operational notion of automation that could be linked to growth and technical change. A notable exception is the new task-based theory of automation proposed by Acemoglu and Restrepo (2018) (AR18, hereafter), but even their insights are limited to a specific rebalancing mechanism featuring productivity-

[1] Karabarbounis and Neiman (2013) measure this decline in multiple countries and conclude that the labor share has been falling globally since the 1980s. They attribute the decline to changing technology. Dao et al. (2017) provide updated evidence. While the decline in the aggregate labor share remains a controversial issue due to measurement challenges (Gutierrez and Piton, 2020; Koh et al., 2020), its decline in heavily automated and automating sectors is indisputable given their magnitude (e.g., US manufacturing).

[2] The constancy of the labor share is one of Kaldor's facts of growth (Kaldor, 1961). The monographs by Brynjolfsson and McAfee (2014), Ford (2009) and Frey (2020) highlight the growing concerns associated with modern automation and provide an anecdotal characterization of the core technologies that it involves. Evidence linking modern automation to labor displacement can be found in Acemoglu and Restrepo (2020), Autor and Salomons (2018), Hubmer (2020), Humlum (2019), and Graetz and Michaels (2018).

enhancing R&D process that replaces the existing and already automated tasks by newly created and initially labor-intensive tasks.

The goal of the current paper is to fill this gap by providing a parsimonious theory of automation that bridges these different strands of the literature and offers generalized answers to the questions listed above. In particular, while building on AR18's work, we depart here from their view of the task space as a dynamic object linked to a specific form of structural change. Instead, the task space is static in our model and thus most appropriately interpreted as the *universe* of *all* feasible operations with matter that are potentially relevant for production. Firms use some of these tasks to produce the current set of goods, and in the background, tasks are assumed to churn between being used, out of use, or reused.

The key simplifying assumption that underlies our approach is that the economy is assumed to be sufficiently large and the time horizon sufficiently long so that the properties of tasks in use become effectively divorced from any specific process of structural change and innovation occurring throughout the economy due to random task churning.[3] This assumption leads to a simpler representation of technical change as an exogenous schedule of evolving productivity of capital across tasks, allowing us to study a broader range of its possible incarnations. It implies that the fully diffused view of uniformly capital-augmenting technical progress across all tasks is a natural benchmark, and features such as the rents to R&D associated with automation innovation decrease in their importance because productivity of capital within tasks becomes a cumulative product of incremental innovations over their long lifetime.[4]

As in AR18, our specification of task technology similarly assumes that there exists a sufficiently fine breakdown of production into a set of basic operations—called *tasks*—which in our case make capital and labor perfectly substitutable at a task-specific fixed ratio. The existence of such a separation defines what *automation* and *automation capital* are, and not all capital needs to exhibit this property. After sorting tasks by that ratio—which we refer to as the *complexity* rank of a task for capital—the usage of automation capital across tasks by a

[3]Our interpretation of the task space implies that even when the R&D process driving automation is directed toward some subset of tasks in the short run, the continual process of random structural changes occurring in the economy tends to reshuffle tasks and diffuse the impact of R&D on capital productivity across tasks, resulting in the effect of R&D on capital productivity being diffused across tasks in the long run.

[4]In models of R&D based on the quality ladders framework (Grossman and Helpman, 1991), rents accruing to R&D automation are determined by the marginal leap in productivity delivered by a new "automation recipe." As a result, in the long run, rents associated with the cumulative innovation that determines the overall productivity of capital up to that point within a task are small, and technical progress can be approximately seen as rent-free. AR18 track rents because new tasks emerge as the economy grows.

cost minimizing firm is determined via a cutoff rule. Automation is associated with any form of technical change that increases the productivity of capital across tasks and that cutoff, and in the paper we explore its general form that nests a broad range of its possible "biased" incarnations to examine the general connection between technical change, automation, and the labor share. Our full model additionally features capital that is specific to tasks, and its production similarly involves the completion of tasks. This endogenizes the notion of task complexity as the *task load* associated with the production of a machine specific to a given task.

The first substantive insight from our theory is that the assumed form of technical change driving automation is crucial for understanding how it affects the labor share; in particular, if technical progress is "diffused" in that it augments the productivity of capital broadly across tasks, automation need not be labor-share displacing. However, if technical progress is *sufficiently* biased toward marginal, first-to-be automated tasks, automation is *necessarily* associated with a decline in the labor share. Intermediate cases fall in between these extremes.

The key to this result is the observation that the displacement effect of automation is solely determined by the way technology affects the price of capital on the margin; that is, its price within tasks that are not yet automated but are first in line to be automated. Since the cost of using labor and automation capital is equal on the margin, the productivity gain associated with automating marginal tasks is *always* nil, implying a negative net effect in that case (a consequence of the envelope theorem). In contrast, if automation is driven by diffused technical change that increases the productivity of capital across a whole spectrum of tasks, including those that are already automated, the productivity effect of automation becomes positive and the net effect can also be positive. The crucial condition for the net effect to be positive is that the average productivity of capital across the so far automated tasks in relation to the marginal (last) automated task is not too high. The reason is that this is what determines the impact of diffused technical change on firm profits, and hence prices after imposing zero profits.

To examine the long-run growth implications of our model, we embed it into the neoclassical growth model. We impose balanced growth restrictions that feature automation and constant factor shares and seek to identify the task technology that is consistent with this requirement. We show that these conditions lead to a unique task technology that aggregates to the Cobb–Douglas production function, a result closely related to the corollary to Uzawa's theorem.[5]

[5]The microfoundation of the Cobb–Douglas production function are a new result. AR18 obtain a Cobb--Douglas production function only under the assumption of unit elasticity between tasks (i.e., Cobb--Douglas

In our full model featuring the production of task-specific capital, this result boils down to the requirement that the power law governs the measure of tasks needed to produce machines that are specific to tasks of a given complexity rank. This is an appealing feature because the power law arises spontaneously in nature and hence this technology has the potential of being nongeneric.[6] The existence of balanced growth shows that there is no contradiction between automation and the Kaldor facts as long as the technical progress affecting the productivity of capital is diffused across tasks, but it also raises a follow-up question regarding why the labor share might be now declining due to modern automation—as hypothesized by the literature.

In this regard, our answer is that IT may be the culprit because, in our framework, it represents a form of "complexity-biased" technical change. Specifically, we use our theory to propose a concrete, albeit stylized, model of how the emergence of IT-based automation adopted by profit-maximizing firms can endogenously lead to such an effect. The key feature of this model is that a capital-producing firm can use IT to "compress" the task load required to produce a machine (capital) at the expense of completing a fixed measure of some other tasks—with the degree of compression being optimally chosen by that firm. The idea is that the fixed set of tasks is associated with adding a computer chip and/or lines of computer code to obtain "smart" machines that optimize the use of hardware. To the extent that the technologies that drive the current wave of automation exhibit the characteristics of the proposed technology, they can be labor share-displacing, and the diffusion argument does not apply because of the specific and universal nature of this enabling-technology. (Section 3 provides a brief discussion of how this view fits into the anecdotal evidence about the role of IT in modern automation.)

To summarize our findings, it is helpful to invoke Leontief's analogy between humans and horses that AR18 reference in their work. Following up on that analogy, they ask: What differentiates humans from horses so that they will not share the fate of becoming redundant in the course of modern automation? AR18's answer is that humans, unlike horses, can create new and initially labor-intensive tasks that crowd out previously automated tasks (possibly associated with the replacement of old goods or production techniques in the economy). In contrast, our paper offers a complementary and less specific answer: the fact that humans are fungible across a vast array of churning feasible tasks as the economy undergoes structural change can

aggregation of tasks) and a fixed automation margin.

[6] While this is not our focus, Online Appendix F discusses how this technology can be further endogenized as a result of a nongeneric and random process of task-level innovations.

be enough of a defense line for labor to maintain its share. Put differently, it is not necessarily "human creativity" but more generally "task churning" that can hold the line for labor by diffusing the effects of the productivity-enhancing innovation in automation technology. Our results do not imply that humans are invincible to technical change. Rather, what we show is that not all forms of technical change that are associated with automation need to be labor-share displacing.

As discussed, our work b is most closely related to AR18 as well as to the existing microfoundations of the workhorse Cobb–Douglas production function by Jones (2005) and Houthakker (1955).[7] Regarding the aforementioned microfoundations for the Cobb–Douglas production function, a notable feature of our task-based microfoundation is that it does not require that aggregation occurs on an economy-wide level to obtain the Cobb–Douglas production function (approximately)—since the cost minimization in the use of capital and labor per task can be dispersed across heterogeneous firms.[8] The shared feature is the Pareto distribution. We do not have a clear intuition for this connection other than the fact that the Pareto distribution appears to deliver the right kind of curvature across structurally different models.

1 Baseline model of production

In this section, we lay out our baseline theory of production in a static partial equilibrium setting and characterize the link between automation and the labor share for a general form of capital-augmenting technical change. The next sections turn this setup into a fully fledged growth model and generalize the notion of technology and task complexity.

1.1 Environment

The basic unit of production is a firm: an abstract optimizing unit representative of the economy as a whole. The firm takes prices as given and produces a homogeneous good sold in

[7] The recent work by Hubmer and Restrepo (2021) is also relevant and complementary in terms of its focus on the firm-level linkages between automation and declines in the firm-level labor share.

[8] The issue is that aggregation requires that firm optimization occurs on the economy-wide level, as discussed in Acemoglu (2009) (Section 15.8, p. 526). In particular, Acemoglu writes: "(...) existing evidence indicates that there are considerable differences in the production function across industries, and they cannot be well approximated by Cobb-Douglas production function. This suggests it would be interesting to combine the aggregation (...) with equilibrium interactions, which might delineate at what level the aggregation should take place and why (...)." We show how to overcome it in Online Appendix C and discuss it in Section 2.2.

a competitive market for price $P > 0$. There are two factors of production: capital and labor. The user cost of capital is $r > 0$ and the wage rate is $w > 0$ (here exogenous). We later discuss how to think about firm heterogeneity, but our baseline setup features a representative firm.

Technology

To produce one unit of output, the firm must complete a measure of tasks indexed on the real line by $q \in \mathcal{Q} = \mathbb{R}_+ := [0, +\infty)$. A task is a basic operation that can be either performed by a unit of labor or $k(q)$ units of capital. There is no substitution between tasks in that completing a subset of tasks many times does not reduce requirement to complete other tasks.[9]

Tasks are sorted by capital requirement, implying that $k(q)$ is increasing. Throughout, we refer to q as task *complexity*.[10] The unit labor requirement is a normalization, and the underlying assumption is that the capital requirement and the labor requirement are i.i.d. across tasks—as we show in Online Appendix A.[11]

The measure of tasks is determined by a measure function $\mu : \mathscr{B}(Q) \to \mathbb{R}_+ \cup \{+\infty\}$, where $\mathscr{B}(Q)$ denotes the Borel σ-algebra over $\mathcal{Q}$. We assume the existence of the derivative of the measure, implying that μ is generated by some nonnegative Lebesgue measurable function g referred to as density (not necessarily probability density); that is, for any Borel subset $\mathcal{S} \subseteq \mathcal{Q}$ of the complexity space, we have[12]

$$\mu(\mathcal{S}) = \int\limits_{\mathcal{S}} g(q)\, dv, \tag{1}$$

where v is the Lebesgue measure of the real line. To simplify the analysis, we assume that g

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[9]The assumption that all tasks must be completed contrasts with related task models that allow for some degree of substitutability between tasks. For a fixed commodity that involves, say, tasks A and B, it is not clear what it physically means that completing task A twice is a substitute for completing task B. On the other hand, features such as a broader technology menu from which firms might be choosing, or differentiated goods, should be modeled explicitly in a microfounded model when such distinctions are critical for analysis. Our goal is to formulate the technology in a way that that does not mix in preferences.

[10]Since it is increasing, $k(q)$ is almost everywhere differentiable and continuous except for a countable number of points.

[11]The setup considered here is equivalent to a setup featuring a variable labor requirement $l(q)$ under the assumption that $k(q)$ and $l(q)$ are independent across tasks. What permits this normalization is the fact that *all* tasks must be completed to produce a unit of output.

[12]The integral of a measurable function over a measure defines another measure—see, for example, Billingsley (1995) Theorem 16.9 (p. 212-213).. By the Radon-Nikodym theorem—which provides conditions for the existence of the inverse of this mapping (obtaining g from a given μ)—the family of measures admitted by this formula includes all measures that are absolutely continuous with respect to the Lebesgue measure.

has full support, which involves little loss in generality.[13] To ensure that production is feasible, we assume there exists at least one measurable partition of the complexity space that involves tasks assigned to each respective factor so that total input usage is finite.

Assumption 1. *There exists a partition $\{\mathcal{Q}_k, \mathcal{Q}_l\}$ of $\mathcal{Q}$ with $\mathcal{Q}_k \cap \mathcal{Q}_l = \emptyset$* and $\mathcal{Q}_k \cup \mathcal{Q}_l = \mathcal{Q}$ *such that $\int_{\mathcal{Q}_l} 1 d\mu < \infty$ and $\int_{\mathcal{Q}_k} k(q)\, d\mu < \infty$, and $k(q)$ is strictly positive on at least part of the domain.*

In summary, the description of technology in our model is a tuple $T := (k, g)$ or, interchangeably, we use $T = (k, \mu)$ or $T = (k, S)$, where S is the survival function associated with g:

$$S(q^*) = \int_{q^*}^{\infty} g(q)\, dq. \tag{2}$$

If T obeys the above assumptions, we say that it is an *admissible task technology* and denote this set by $\mathcal{T}$. If, in addition, g can be normalized to a probability density function (pdf), we say that technology T has a probabilistic representation.

Firm problem

Production technology exhibits constant returns to scale: to produce Y units of output the firm repeats the tasks needed to produce one unit of output Y times, where the underlying unit is sufficiently small to warrant $Y \in \mathbb{R}_+$. Therefore, the profit maximizing firm chooses output level $Y > 0$ (scale) to maximize its flow profits given by $\Pi = PY - c(w, r)\, Y$, where PY is revenue, $c(w, r)\, Y$ is total cost, and $c(w, r)$ is both the marginal cost and the unit cost. The profit maximization problem under constant returns to scale is linear and hence ill-defined unless $P = c(w, r)$, in which case the firm is indifferent to the choice of output Y. Throughout, we assume that the output price P is such that profits are zero and Y clears the market.

The central element of the firm's optimization problem is the cost minimization problem that defines the unit/marginal cost $c(w, r)$. We need additional notation to define this cost. Let $\mathscr{P}$ be the collection of all partitions of the complexity space $\mathcal{Q}$ to measurable subsets $\mathcal{Q}_k, \mathcal{Q}_l$

[13] The set on which $g(q) = 0$ can be eliminated from the domain with no impact on production (since inputs are zero). The space can be stretched to fill the real line. Irregular cases such as "fat" Cantor sets are of no economic relevance here.

such that $\mathcal{Q}_k \cap \mathcal{Q}_l = \emptyset$ and $\mathcal{Q}_k \cup \mathcal{Q}_l = \mathcal{Q}$. The unit cost then solves

$$c\left(w, r\right) := \min_{K, L, \{\mathcal{Q}_k, \mathcal{Q}_l\} \in \mathscr{P}} rK + wL \tag{3}$$

subject to two constraints underlying the task technology:

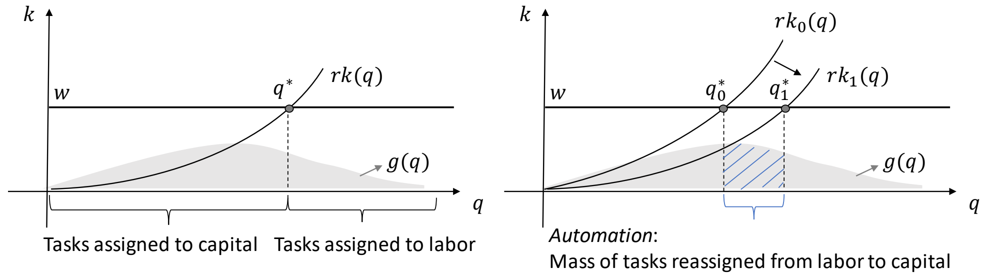
$$L = \int_{\mathcal{Q}_l} 1 d\mu = \mu\left(\mathcal{Q}_l\right) \text{ and } K = \int_{\mathcal{Q}_k} k\left(q\right) d\mu = \int_{\mathcal{Q}_k} k\left(q\right) g\left(q\right) dv. \tag{4}$$

The first constraint states that labor usage, L, is determined by the measure of the tasks assigned to labor, which is $\mu\left(\mathcal{Q}_l\right)$ due to the unit normalization of the labor requirement per task. The second constraint states that capital usage, K, is determined by the capital requirement function $k\left(q\right)$ integrated over the measure μ on the set $\mathcal{Q}_k$ of tasks assigned to capital, or equivalently, by the product $k\left(q\right) g\left(q\right)$ integrated over the Lebesgue measure v. We previously assumed that there exists a partition that makes production feasible, so the cost minimization problem above is well defined. For later use, we denote the optimal factor intensity implied by the solution(s) to (3) as $\frac{K}{Y}\left(\frac{w}{r}\right)$ and $\frac{L}{Y}\left(\frac{w}{r}\right)$, and refer to them throughout as *capital intensity* and *labor intensity*, respectively. (At this point, these can be either functions or correspondences.)

We next establish that the solution to the firm's cost minimization problem amounts to finding a cutoff value $q^* \geq 0$ that partitions the complexity space in such a way that all tasks below q^* are completed using capital and all tasks above q^* are completed using labor. The optimality of a cutoff rule is intuitive, but obtaining this result requires that the integrals underlying the constraints of this problem are finite under a cutoff rule and Assumption 1—which we show in Lemma 1 below. (All proofs are in the appendix at the end unless otherwise noted.)

Lemma 1. *Cost minimization in* (3) *given* $T \in \mathcal{T}$ *involves a cutoff rule such that tasks on the interval* $[0, q^*]$ *are completed using capital and the remaining tasks are completed using labor, where* $q^* \in \mathbb{R}_+ \cup \{+\infty\}$ *is such that: i) for any* $0 < q^* < \infty$ *there exists* $\varepsilon > 0$ *such that for all* $0 < \delta < \varepsilon$, $rk\left(q^* - \delta\right) \leq w$ *and* $rk\left(q^* + \delta\right) \geq w$*, or else* $q^* = 0$ *and* $rk\left(q^*\right) \geq w$*, or* $q^* = \infty$ *(up to a sets of measure zero).*

At the cutoff point, the firm is indifferent to which factor it uses, and so except for countably many points of discontinuity and irregular cases such as k bounded from above, the cutoff q^* satisfies the indifference condition $rk\left(q^*\right) = w$. The solution is unique as long as k is strictly



(a)

(b)

Figure 1: Firm cost minimization problem (a) and the notion of automation (b).

Notes: Panel a illustrates the cost–minimizing division of tasks into those assigned to capital and those assigned to labor. The task complexity rank is on the horizontal axis, and the cost of completing a task is on the vertical axis. The shaded area shows the density g of tasks that must be completed. The $rk(q)$ schedule is the cost of completing a task using capital and the w schedule is the cost of completing a task using labor. The cost–minimizing partition is given by cutoff q^*, where the two schedules intersect, with the tasks to the left of this cutoff point being automated. Panel b illustrates the comparative statics associated with a downward shift in the capital requirement function k. As shown, this leads to the *automation* of tasks in the striped region.

increasing in the neighborhood of q^*. Otherwise, there can be a range of values. We refer to this cutoff as the *automation cutoff*.

Figure 1a illustrates the firm's cost minimization problem graphically. Task complexity q is on the horizontal axis and the factor cost of completing each task is on the vertical axis. The boundary of the shaded area depicts task density g. The cost of completing tasks using capital is given by the rk schedule and the cost of completing tasks using labor is given by the w schedule. To minimize costs, the firm uses capital to the left of the cutoff q^* where the cost of completing tasks using capital (rk schedule) intersects the cost of completing tasks using labor (w schedule). We refer to these tasks as *automated tasks*, since the remaining tasks are completed using labor. We say that *automation* occurs whenever there is an increase in the share of tasks assigned to capital. As an example, Figure 1b depicts the impact of a downward shift in rk from its initial position rk_0 to a new position denoted by rk_1. As shown, this leads to an increase in the cutoff and the automation of a mass of tasks corresponding to the striped area in the figure.

Aggregation of task technology

To examine the connection between automation, technical change driving automation, and the labor share, it is convenient to introduce a notion of a production function. A production function of a firm is the maximum output Y that can be produced from fixed (aggregate) inputs

K and L:

$$Y(T;K,L) := \sup\left\{\hat{Y}\in\mathbb{R}_+ : \exists_{q^*\in\mathcal{Q}} \text{ s.t. } K \geq \hat{Y}\int_0^{q^*} k(q)\,d\mu, L \geq \hat{Y}\int_{q^*}^{\infty} 1d\mu\right\}, \tag{5}$$

where $T=\{k,\mu\}$, and here it pertains to the economy as a whole. The monotonicity of capital requirement function $k(q)$ and the assumption that it is nonzero on at least part of the domain $\mathcal{Q}$ implies that determining $Y(T;K,L)$ for a given tuple K,L amounts to choosing a *factor utilization cutoff* q^* so that all inputs are *fully* utilized; that is, the technological constraints in (5) hold with *equality*. Note this cutoff is different from the automation cutoff implied by firm's cost minimization problem above, which is defined on the price domain $\frac{w}{r}$ and features a free choice of factor intensities based on the relative price of these inputs—albeit in equilibrium that clears factor markets they will be equal. We summarize our results in the lemma below.

Lemma 2. *For any inputs $K>0, L>0$ and given technology given $T=\{k,g\}\in\mathcal{T}$, there exists a unique factor utilization cutoff $q^*>0$ and $\hat{Y}>0$ such that $K=\hat{Y}\int_0^{q^*}k(q)\,d\mu$, $L=\hat{Y}\int_{q^*}^{\infty}d\mu$, and $Y(T;K,L)=\hat{Y}$, where $Y(T;K,L)$* is defined by (5).

Our last result shows that the marginal products are well defined and we derive the formula for output elasticity with respect to each factor. In a competitive market environment output elasticities map onto factor shares, and this formula is of interest.

Lemma 3. *Marginal products $MPK := \frac{\partial Y(T;K,L)}{\partial K}$, $MPL := \frac{\partial Y(T;K,L)}{\partial L}$ implied by $Y(T;K,L)$* in (5) *are well defined almost everywhere and the implied output elasticity with respect to $K>0$ and $L>0$ is*

$$\alpha := \frac{K}{Y}MPK = \left(1+\frac{k(q^*)}{K}L\right)^{-1} \text{ and } \frac{L}{Y}MPL = 1-\alpha, \tag{6}$$

respectively, where q^* is the factor utilization cutoff satisfying the conditions of Lemma 2.

1.2 Automation and the labor share

We begin our analysis by characterizing the link between automation and the labor share for a general form of capital-augmenting technical progress. We use these results to guide our analysis throughout the rest of the paper.

Definition of $v(q)$-generalized marginal capital-augmenting technical progress

To represent capital-augmenting technological change as generally as possible, we consider a perturbation of the capital requirement function of the form: $k_\varepsilon(q) = k_0(q) - \varepsilon v(q)$, where $k_0(q)$ is the initial level, $v(q)$ is a nonnegative and differentiable function, and ε is a nonnegative scalar determining the degree of perturbation. In what follows, we focus on the marginal impact of ε at $\varepsilon = 0$ in the spirit of variational calculus and characterize how it depends on the properties of $k_0(q)$ and $v(q)$. We refer to this form of technical progress as $v(q)$-*generalized marginal capital-augmenting technical progress*. We do not separately treat technical change that affects task density $g(q)$, but its effects will be clear from the discussion that follows.

Prices r, w are held constant in this exercise but not the price level P, which adjusts so that profits of the firm are zero at all times. The adjustment of the price level in the course of this variation is key to redistributing the gains from automation between the factors of production and assessing the impact of automation and technical change on the labor share in a way that is consistent with the notion of equilibrium in our full model (as laid out in Section 3).

The effect of such a perturbation in a "nonmarginal form" of $\varepsilon > 0$ is conceptually illustrated in Figure 1b. The figure depicts a (nonmarginal) shift in the capital requirement function from its initial position, labeled $k_0(q)$, to some new position $k_1(q)$, which, for example, could be associated with $v(q) \equiv k_0(q)$ for some fixed $\varepsilon > 0$ in order to define $k_1(q)$. As already discussed, labor intensity $\frac{L}{Y}\left(\frac{w}{r}\right)$ decreases and capital intensity $\frac{K}{Y}\left(\frac{w}{r}\right)$ increases due to the automation of the striped mass of tasks labeled in the figure. Firm profits initially rise, and the price P must decline because profits are required to be zero. Importantly, the decline in P does affect the cutoff because it depends on $\frac{w}{r}$.

Results

The labor share under this variation is given by the expression

$$LS_\varepsilon := \frac{w}{P_\varepsilon}\frac{L}{Y}\left(q_\varepsilon^*\right), \tag{7}$$

where $\frac{L}{Y}(q)$ is the labor requirement function that minimizes production cost and q_ε^* is the cost minimizing automation cutoff. Differentiating this expression with respect to ε (at $\varepsilon = 0$) yields a decomposition of the resulting (relative) change of the labor share:

$$\frac{d\log LS_{\varepsilon}}{d\varepsilon}|_{\varepsilon=0}=\underbrace{\frac{d\log\frac{L}{Y}\left(q_{\varepsilon}^{*}\right)}{d\varepsilon}|_{\varepsilon=0}}_{\text{displacement effect DE}}+\underbrace{-\frac{d\log P_{\varepsilon}}{d\varepsilon}|_{\varepsilon=0}}_{\text{productivity effect PE}}. \tag{8}$$

As indicated, it comprises the *displacement effect* (DE) and the *productivity effect* (PE).

The displacement effect is associated with the displacement of labor by capital on the margin—conceptually represented by the striped area in Figure 1b. The automation cutoff—assuming that $k_0\left(q\right)$ is strictly increasing and differentiable at q_0^*—satisfies the identify $k_0\left(q_{\varepsilon}^*\right)-\varepsilon v\left(q_{\varepsilon}^*\right)\equiv\frac{w}{r}$, and hence

$$\frac{dq_{\varepsilon}^{*}}{d\varepsilon}|_{\varepsilon=0}=\frac{v\left(q_{0}^{*}\right)}{k'\left(q_{0}^{*}\right)}>0, \tag{9}$$

where $k'\left(q\right)$ is shorthand for the derivative of $k\left(q\right)$. As expected, the expression is nonnegative, and as long as technical progress has a marginal effect, that is, $v\left(q_0^*\right)>0$, it is strictly positive. The equation shows that the increase in the cutoff depends on both the marginal impact of technical change as well as the (local) steepness of the capital requirement schedule at q_0^*.

The productivity effect is associated with a decline in the zero-profit price P. Differentiating the zero-profit condition of the firm $\left(w\frac{L}{Y}\left(q_{\varepsilon}^*\right)+r\frac{K}{Y}\left(q_{\varepsilon}^*\right)-P_{\varepsilon}\equiv 0\right)$ shows that the productivity effect is generally associated with the impact of automation as well as the direct impact of the technical change driving the automation:

$$\frac{dP_{\varepsilon}}{d\varepsilon}|_{\varepsilon=0}=\underbrace{w\frac{d\frac{L}{Y}\left(q_{\varepsilon}^{*}\right)}{d\varepsilon}|_{\varepsilon=0}+r\frac{d\frac{K}{Y}\left(q_{\varepsilon}^{*}\right)}{d\varepsilon}|_{\varepsilon=0}}_{\text{effect of automation}}-\underbrace{v\left(q_{0}^{*}\right)}_{\text{direct effect of technical change}}, \tag{10}$$

The productivity effect is nonnegative, since we are focusing here on the case of increased productivity of capital. The relative magnitude of these effects is central for the link between automation and the labor share.

The proposition below establishes the necessary and sufficient conditions for the net effect to be positive. We discuss these conditions next and lay out the key intuitions behind them.

Proposition 1. *$v(q)$-generalized marginal capital-augmenting technical progress implies: 1) the rate of change in the tasks operated by labor is*

$$\frac{d\log S\left(q_{\varepsilon}^{*}\right)}{d\varepsilon}|_{\varepsilon=1}=-h\left(q_{0}^{*}\right)\frac{v\left(q_{0}^{*}\right)}{k'\left(q_{0}^{*}\right)}, \tag{11}$$

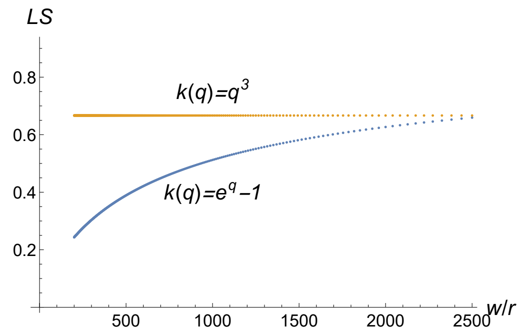


Figure 2: The effect of uniform capital-augmenting technical progress (higher w/r) on the labor share.

Notes: The figure plots the trace of the labor share implied by the firm problem with respect to w/r in an exercise that assumes an exogenous decrease in r, fixed w, and zero profit P. The first case features $k = q^3$. The second case features a less convex function given by $k(q) = e^1 - 1$. In both cases, we assume Pareto pdf $g(q) \propto (q + .1)^{-2}$. Automation—corresponding to an increase in the assumed w/r ratio—results in an increase in the labor share under the $k(q)$ schedule in the second case, while the labor share is flat in the first case.

and 2) the rate of change in the labor's share in income is

$$\frac{d \log LS_\varepsilon}{d\varepsilon}|_{\varepsilon=0} = \underbrace{\frac{d \log \frac{L}{Y}\left(q_\varepsilon^*\right)}{d\varepsilon}|_{\varepsilon=0}}_{\textit{displacement effect DE}} + \underbrace{-\frac{d \log P_\varepsilon}{d\varepsilon}|_{\varepsilon=0}}_{\textit{productivity effect PE}}, \tag{12}$$

where

$$DE = \frac{d \log S\left(q_\varepsilon^*\right)}{d\varepsilon}|_{\varepsilon=1} = -h\left(q_0^*\right)\frac{v\left(q_0^*\right)}{k_0'\left(q_0^*\right)},$$

$$PE = \frac{r}{P_0}\int_0^{q_0^*} v\left(q\right) g\left(q\right) dq = h\left(q_0^*\right) LS_0 \frac{\int_0^{q^*} v\left(q\right) g\left(q\right) dq}{k_0\left(q_0^*\right) g\left(q_0^*\right)},$$

q_0^* is the automation cutoff such that at that cutoff point $k_0(q)$ is strictly increasing and differentiable, and where $h(q) := -\frac{dS(q)}{dq} = \frac{g(q)}{S(q)}$ is the hazard rate associated to g. (Almost everywhere on $\mathcal{Q}$.)

The first key implication of the above proposition is that technical progress must be diffused and must augment the productivity of capital across a broader range of tasks for automation to have a chance of being associated with a nonnegative impact on the labor share in (12). This is because—by the envelope theorem associated with the optimality of the cutoff—the productivity effect is solely attributable to the direct effect of technical change as labeled in (10); that is, the effect of automation is nil. As a result, any *marginal* form of technical progress featuring $v\left(q_0^*\right) = k_0\left(q_0^*\right)$ and vanishing $v(q) \approx 0$ over the range of currently automated tasks on the interval $[0, q^*)$ entails only the negative displacement effect. This is easily verified by plugging in these values into (12) and noting $PE = 0$ and $DE < 0$.

The second key implication is that diffused or nonmarginal nature of technical progress is not enough and the technology in place must be favorable to enhance the productivity effect ($PE + DE \geq 0$). Let us discuss what conditions on technology are conducive to such an outcome.

To that end, consider the standard case of *uniformly capital-augmenting* progress driven by a decline in the price of capital goods; that is, let $v(q) \equiv k_0(q)$, which conceptually corresponds to what is illustrated in Figure 1b. After replacing v with k in (12), the productivity effect is:

$$PE = \frac{r}{P_0} \int_0^{q_0^*} k_0(q)\, g(q)\, dq = h(q_0^*)\, LS_0 \frac{\int_0^{q_0^*} k_0(q)\, g(q)\, dq}{k_0(q_0^*)\, g(q_0^*)}.$$

The first equality shows that the productivity effect is associated with the reduction in the firm's spending on capital due to the increased productivity of capital across the already automated tasks on the interval $[0, q^*)$—which in our model fully accrues to labor income under zero profits because the model features no substitution of one task for another task (that is, the technology is Leontief as far as the tasks go). Had there been any substitution between tasks, the result would depend on the corresponding elasticity. However, for reasons discussed in footnote 9, we favor such a formulation of technology. The second equality shows that the initial share of labor and the average productivity of capital across the automated task relative to the marginal task are decisive. This equality implies that a more convex $k(q)\, g(q)$ schedule is conducive to generating a larger productivity effect (for a fixed value of the labor share LS_0^*).

The strength of the displacement effect confirms that convexity of this schedule leads to the positive net effect, as it is given by

$$DE = \frac{d \log S(q_\varepsilon^*)}{d\varepsilon}\Big|_{\varepsilon=1} = -h(q_0^*)\, \frac{k_0(q_0^*)}{k_0'(q_0^*)}.$$

The first equality shows that displacement corresponds to the mass of tasks completed by labor—which is intuitive given that the labor requirement per task has been normalized to unity. The second equality shows that the more convex k schedule is, the smaller the change in the cutoff, and the smaller the displacement effect. The hazard rate is unimportant because it scales both the productivity effect and the displacement effect.[14]

Can these conditions be ever satisfied? Figure 2 depicts results implied by two numerical examples. The first example features $k(q) = q^3$ and the second example features a less convex schedule $k(q) = e^q - 1$. In both cases, we assume that task density is given by the Pareto probability density function (pdf) $g(q) \propto (q + .1)^{-2}$, and we solve for the equilibrium for a range of values for w/r (and zero profit P). As we can see, a higher $\frac{w}{r}$—which is associated with an increase in the automation cutoff—implies a flat path for the labor share in the first case and an increasing path in the second case. The answer is affirmative.

Concluding, the key lesson from the above analysis is that both the technology in place and the form of technical progress that drive automation affect the labor share. In particular, observations of automation alone are insufficient to determine its effects on the labor share. As a general rule, technical progress that is more diffused and improves the (relative) productivity of capital across a vast swath of tasks tends to boost the productivity effect. The net effect then depends on how productive capital is over the entire range of automated tasks vis-à-vis the marginal automated task. Labor must hold its own on that range and capital cannot be "too" productive to sustain the labor share. As discussed in the introduction, the assumptions underlying the notion of tasks in our model favor a diffused view of technical progress in the long run unless there is a specific reason to think otherwise. Any random churning of tasks that occurs in the economy will tend to diffuse the impact of even extremely marginal progress over time, and for this reason there is hope for labor to hold the line as long as it remains reasonably productive on the range of the automated tasks relative to the marginal task. Positive rents to automation-enabling R&D could be a countervailing force as they (predominantly) accrue to capital, but our model assumes them away for the reasons discussed above.[15]

[14]Note that $h(q_0^*) := \frac{g(q_0^*)}{S(q_0^*)} = -\frac{d \log S(q_0^*)}{dq(q_0^*)}$.

[15]In the medium run, and hence in the context of the recent declines in the aggregate labor share that span just a few decades, rents to R&D associated with automation technology could be a factor. In fact, one of the hypotheses for the declining labor share in the aggregate data is that the rents associated with intangible capital have increased (Koh et al., 2020).

2 Balanced growth with automation

The next question we ask is whether automation is consistent with past growth experiences, namely the Kaldor facts (Kaldor, 1961).[16] This requires that automation occurs as the economy grows and yet the labor share is constant and growth is balanced. We show that there is no contradiction between these observation as long as technical progress is uniformly capital-augmenting, and the next section explores the question of what happens when it is not.

2.1 Growth model and general equilibrium

The overarching growth model is standard but we describe the setup to lay out notation. Time is continuous, $t \in \mathbb{R}_+$ and $T_t \in \mathcal{T}$ denotes the exogenous path for the evaluation technology in the economy (under perfect foresight), which we summarize by production function $Y\left(T_t; K_t, \bar{L}\right)$ defined by (5).[17] The economy is populated by a stand-in firm and a stand-in household. The household values consumption streams according to the present discounted value of the flow utility from consumption C_t given by $u\left(C_t\right) = \frac{C_t^{1-\sigma}}{1-\sigma}$, $\sigma > 1$, where $0 < \rho < 1$ denotes the discount factor. For simplicity, the household inelastically supplies $\bar{L}$ units of labor and accumulates capital that it rents to firms. All markets are perfectly competitive and all prices are taken as given.

The *allocation* corresponds to any non-negative path of C_t, K_t, and Y_t, as well as factor utilization cutoff q_t^* associated with $Y\left(T_t; K_t, \bar{L}\right)$ as in Lemma 3. The allocation must obey the economy-wide resource constraint

$$C_t + \dot{K}_t - \delta K_t = Y_t = Y\left(T_t; K_t, \bar{L}\right), \tag{13}$$

where $C_t \geq 0$ is consumption (in period t), $K_t \geq 0$ is capital used in production, $\dot{K}_t := \frac{dK_t}{dt}$ is the gross investment in capital, δK_t is the depreciation of capital, and $\bar{L}$ is labor used to production. By welfare theorems, the *equilibrium allocation* solves the planning problem of maximizing lifetime utility $\int_0^\infty e^{-\rho t} u\left(C_t\right)$ subject to the resource constraint in (13) and given $K_0 > 0$.[18]

[16] For a review of the Kaldor facts, see Jones and Romer (2010).

[17] By a path we mean a function of time t.

[18] As for the *competitive equilibrium*, the aggregates C_t, K_t, Y_t are determined by the household who receives income $w_t\bar{L} + r_t K_t$, faces a budget constraint $P_t\left(C_t + \dot{K}_t - K_t + \delta K_t\right) = w_t\bar{L} + r_t K_t$. Given $K_0 > 0$, the

2.2 Conditions for balanced growth with automation

In a model with detailed microfoundations, the definition of a balanced growth path (BGP) is more involved because it has to specify how various functional forms evolve over time. We use the standard approach of assuming "stable shape" conditions in the spirit of similar definitions used by, for example, Lucas and Moll (2014), Perla and Tonetti (2014), and Menzio and Martellini (2020). These assumptions are not without loss, but they are justified. The logic is that the underlying endogenous processes that determine these objects are stationary along the balanced growth path in the sense that they lead to stable structural relationships within the economy.

Specifically, our definition of BGP requires that the growth rate of the capital requirement per task is independent of the task by assuming that $k(q)$ grows at a constant rate for all $q \in \mathcal{Q}$. We impose a similar condition on the density function $g(q)$. Under this definition, note, growth can only be driven by technical progress that is uniformly factor-augmenting across tasks. As we have argued in the introduction, this view is consistent with the long-run perspective and the notion of task space in our model absent any correlating factor that directs capital-augmenting progress toward a particular subset of tasks. The formal definition of BGP is as follows:

Definition 1. A *balanced growth path with automation and constant factor shares* (BGP, hereafter) comprises allocation sequences $Y_t = Y_0 e^{\gamma_Y t}$, $C_t = C_0 e^{\gamma_C t}$, $K_t = K_0 e^{\gamma_K t}$, $q_t^* = q_0^* e^{\gamma_{q^*} t}$ and technology sequence $T_t = \{k_t(q) = k_0(q) e^{\gamma_k t}, g_t(q) = g_0(q) e^{\gamma_g t}\} \in \mathcal{T}$ such that $\gamma_Y > 0$,$\gamma_{q^*} > 0$ and $\alpha \equiv \frac{K_t}{Y_t} MPK_t$ is constant, where γ_*, $Y_0 > 0$, $C_0 > 0$, $K_0 > 0$, $q_0^* > 0$ are some scalars, and $g_0(q)$, $k_0(q)$ are some positive-valued functions.

Our first result, summarized in Proposition 2, shows that BGP as defined above exists and requires task technology of the form:

$$T_0 = \left\{k_0(q) = k_0 q^{\theta}, g_0(q) = g_0 q^{-\zeta-1}\right\}, \tag{14}$$

where, abusing notation, k_0, g_0 are scalars involved in the specification of functions $k_0(q)$, $g_0(q)$. We refer to this task technology as the *BGP task technology*.

household chooses the paths of C_t, K_t to maximize lifetime utility $\int_0^\infty e^{-\rho t} u(C_t)$. The intertemporal condition associated with the household problem is $\sigma \frac{\dot{C}_t}{C_t} = \frac{r_t}{P_t} - \delta - \rho$, and the market clearing requires that firms employ $\bar{L}$ units of labor and rent K_t units of capital, implying $r_t/P_t = MPK_t$ and $w_t/P_t = MPL_t$, where MPK_t, and MPL_t are given by Lemma 3.

The proof follows from the results obtained so far and the basic property that capital K and output Y grow at the same rate $\gamma := \gamma_K = \gamma_Y$ in the overarching growth model—which is an intermediate step in the proof of the Uzawa steady-state growth theorem as found in, for example, Jones and Scrimgeour (2008). Specifically, by (6), for the factor shares to remain constant, $k_t(q_t^*) = k_0(q_0^* e^{\gamma_{q^*} t}) e^{\gamma_k t}$ must grow at the same rate γ, and so $k_0(q)$ must exhibit a constant elasticity with respect to q because q_t^* is also required to grow at a strictly positive rate $\gamma_{g^*} > 0$.[19] Accordingly, we must have $k_0(q) = k_0 q^\theta$ for some $\theta > 0$. Since the total labor supply is fixed at $\bar{L}$ and the resource feasibility requires $\bar{L} = Y_t \int_{q_t^*}^\infty g_t(q)\, dv$, it must be true that $\int_{q_t^*}^\infty g_t(q)\, dv$ declines at a constant rate γ to offset the constant growth of Y_t at rate rate γ. As we show in the appendix, this yields $g_0(q) = g_0 q^{-\zeta-1}$, as stated.[20] (The omitted steps are in the appendix.)

Proposition 2. *If* $\gamma_k < 0$ and $\gamma_g - \alpha\gamma_k > 0$, *BGP exists and features: 1)* $T_0 = \left\{k_0(q) = k_0 q^\theta, g_0(q) = g_0 q^{-\zeta-1}\right\}$, *where* $\zeta := \frac{\gamma_Y + \gamma_g}{\gamma_{q^*}}$, $\theta := \frac{\gamma_Y - \gamma_k}{\gamma_{q^*}}$*; 2)* $\gamma := \gamma_Y = \gamma_C = \gamma_g - \alpha\gamma_k$.

The next result, summarized in Corollary 1, shows that the BGP technology aggregates to the familiar constant returns to scale Cobb–Douglas (CD) production function—which is not surprising given the Uzawa steady-state growth theorem implies that capital-augmenting technical progress in the overarching neoclassical growth model can only be consistent with that aggregate production function, and that is what the above result requires.

To see this connection more clearly, recall that the Uzawa theorem implies that it must be possible to represent the technical progress driving growth as labor augmenting; that is, it must be possible to represent the production function in the neighborhood of the balanced growth path as $Y\left(T_0; K_t, a_t\bar{L}\right)$, for some constant growth path a_t referred to as labor-augmenting technical progress.[21] To obtain the CD production function from this result, more restrictions must be placed on the underlying balanced growth path. For example, imposing the additional condition that the relative price of capital goods is declining at a constant rate along the balanced growth

[19] Constant growth of $k(q^*)$ given constant growth of q^* implies $\frac{dk_0(q^*)}{dq^*}/k(q) = \theta$, for some constant $\theta > 0$, which solves to $k_0(q) = k_0 q^\theta$, for any constant $k_0 > 0$.

[20] $\theta > \zeta$ is required for the integrals to exist.

[21] We use the shorthand notation $\dot{x} := \frac{dx}{dt}$ throughout. For a detailed discussion of the Uzawa theorem and its proof see Jones and Scrimgeour (2004) and Jones and Scrimgeour (2008), or Acemoglu (2009), Theorem 2.6 (p. 60) and Theorem 2.7 (p. 63).

path suffices.[22] In our model, this condition is replaced by the qualitative requirement that there is a steady rate of automation.

Corollary 1. *The aggregate production function along the BGP is Cobb–Douglas; that is, output Y_t and marginal products MPK_t and MPL_t are consistent with those implied by*

$$Y_t(K,L) = A_t (Z_t K)^{\alpha} L^{1-\alpha} \tag{15}$$

for $0 < \alpha = \frac{\zeta}{\theta} < 1$ and constant growth paths A_t, Z_t.

The BGP task technology is the *only* task technology that aggregates to the CD production function up to monotone transformation of the complexity space $\mathcal{Q}$—which is a degree of freedom because q is a rank index and has no physical meaning in the model. However, for the same reason, since this degree of freedom does not have any physical meaning in the model, it can be largely ignored. The reason why we obtained a unique task technology by requiring BGP is because we have imposed the condition that q^* grows at a constant rate.[23] Without this requirement, growth would not be balanced or we would need capital-augmenting progress to be of varying speed to exactly match the shape of this function, and so such a requirement is indeed needed to yield a reasonable definition of BGP.

Definition 2 and Proposition 3 formalize this result by introducing the notion of an equivalence class between technologies and showing that there is a unique class of equivalent technologies in the sense of Definition 2 that aggregate to the same CD production function.

Definition 2. Task technology $\hat{T} = \left(\hat{k}(q), \hat{S}(q)\right) \in \mathcal{T}$ is *essentially equivalent* to task technology $T = (\mathcal{Q}, k(q), S(q)) \in \mathcal{T}$ if there exist a strictly increasing function $f : \mathcal{Q} \to \mathcal{Q}$ such that $\left(C_k \hat{k}(f(q)), C_S \hat{S}(f(q))\right) \equiv T$, for some scalars $C_k > 0, C_S > 0$.

Proposition 3. *Suppose that a task technology $T = (k(q), g(q)) \in \mathcal{T}$ aggregates to a Cobb-Douglas production function $Y(K,L) = A(ZK)^{\alpha} L^{1-\alpha}$ for some scalars $0 < \alpha < 1$, $A > 0$, and $Z > 0$. Then, 1) the measure μ associated with g via (1) is an infinite measure (i.e., $\mu(\mathcal{Q}) = \infty$), and 2) T is essentially equivalent to the canonical Cobb-Douglas task technology*

$$T^{CD} = \left\{ k(q) = Z^{-1} q^{\frac{1}{\alpha}} \frac{1-\alpha}{\alpha}, g(q) = A^{-1} q^{-2} \right\} \tag{16}$$

[22] We lack a good reference for this result and the outline of the proof can be found in Online Appendix G..

[23] The restriction of this result to $\mathcal{T}$ can be generalized but we do not pursue such a generalization to simplify the exposition.

that aggregates to the same production function.

It is not difficult to verify that the monotone transformation $f(q) = \zeta^{-\frac{1}{\zeta}} q^{\frac{1}{\zeta}}$ applied to the BGP technology in (14) indeed yields T^{CD} in (16) (which show explicitly in the example below). We refer to T^{CD} as the *canonical CD task technology* because it directly maps onto the parameters of the CD production function in (15). Example 1 below illustrates the calculations underlying the derivation of the CD production function in our setup and the cutoffs.

Example 1. *We first show that the canonical CD technology in* (16) *aggregates to the CD production function of the form stated in Proposition 3 (and* Corollary 1*). To that end, we use Lemma 2 to calculate*

$$\frac{L}{Y} = \int_{q^*}^{\infty} g(q)\, dq = A^{-1} q^{*-1}, \ \frac{K}{Y} = \int_{0}^{q^*} k(q)\, g(q)\, dq = (AZ)^{-1} q^{*\frac{1}{\alpha}-1}. \tag{17}$$

We eliminate q^* and immediately obtain $Y = A(ZK)^{\alpha} L^{1-\alpha}$, as claimed. The factor utilization cutoff as a function of $\frac{L}{K}$—as in the definition of BGP—is given by $q^*\left(\frac{L}{K}\right) = Z^{\alpha}\left(\frac{L}{K}\right)^{-\alpha}$, while the automation cutoff is $q^*\left(\frac{w}{r}\right) = \left(Z\frac{\alpha}{1-\alpha}\frac{w}{r}\right)^{\alpha}$. The two cutoffs are equal in equilibrium because factor markets must clear, but conceptually they are different. While utilization cutoff is always unique, the automation cutoff need not be unique—unless $k(q)$ is continuous and strictly increasing as is the case here. Note that the proof of Corollary 1 in the appendix shows that the BGP technology T_0 in Proposition 2 aggregates to the CD function of the form: $Y = \frac{1}{g_0}\left(\zeta^{\frac{\theta}{\zeta}-1}\frac{\theta-\zeta}{k_0}K\right)^{\frac{\zeta}{\theta}} L^{1-\frac{\zeta}{\theta}}$. Accordingly, both technologies give rise to the same production function for matching parameters: $\alpha = \frac{\zeta}{\theta}$, $A = \frac{1}{g_0}$ and $Z = \zeta^{\frac{\theta}{\zeta}-1}\frac{\theta-\zeta}{k_0}$. The survival function associated with BGP technology is $\hat{S}(q) = g_0\frac{1}{\zeta}q^{-\zeta}$. Using the transformation of the complexity space by $f(q) = \zeta^{-\frac{1}{\zeta}}q^{\frac{1}{\zeta}}$, as noted in text, after applying the transformation according to Definition 2, gives $S(q) = \hat{S}\left(\zeta^{-\frac{1}{\zeta}}q^{\frac{1}{\zeta}}\right) = g_0 q^{-1}$, and the implied density is $g(q) := -\frac{d}{dq}S(q) = A^{-1}q^{-2}$.

We conclude this section by stating a technical result implied by directly imposing the requirement that the task technology must aggregate to a Cobb-Douglas production function. This provides an alternative route of obtaining the above results without going through a definition of balanced growth and by instead invoking the Uzawa theorem—although without specifying the conditions on the evolution of economic fundamentals. As we can see, k must be proportional to the hazard rate implied by g, and the range of k must cover the entire real line. As shown in the proof of Proposition 3, this combination of properties implies the measure μ is an infinite measure (i.e., $\mu(\mathcal{Q}) = \infty$). We comment on this property in the next section as it is potentially problematic.

Lemma 4. *The aggregate production function is Cobb-Douglas with exponent $0 < \alpha < 1$ if task technology $T = (k, g) \in \mathcal{T}$ satisfies*

$$\int_0^{q^*} rk(q)\, g(q)\, dq \equiv \frac{\alpha}{1-\alpha} rk(q^*)\, S(q^*), \tag{18}$$

almost everywhere, or equivalently k, g solve

$$\alpha \frac{k'(q)}{k(q)} = h(q) := \frac{g(q)}{S(q)}, \tag{19}$$

with $k(q) \to_{q\to 0} 0$, $k(q) \to_{q\to\infty} \infty$, and $S(q^*) := \int_{q^*}^{\infty} g(q)\, dq < \infty$. Accordingly, for all $\varepsilon > 0$ there exists a scalar $C_\varepsilon > 0$ such that for all $q \geq \varepsilon$ $k(q) = C_\varepsilon S(\varepsilon)^{\frac{1}{\alpha}} S(q)^{-\frac{1}{\alpha}}$.

Properties of Cobb–Douglas task technology

The Cobb–Douglas task technology exhibits two key properties. First, the capital requirement per task exhibits power law tails with index α—which is easy to see by evaluating the conditional probability of capital requirement above some fixed value: $p(k \geq k^* | k \geq k_0^*) = \left(\frac{k^*}{k_0^*}\right)^{-\alpha}$. This is true for any CD task technology—not only the canonical CD task technology in (16)—and it is an appealing property because the power law arises spontaneously in nature, implying that the CD task technology gives hope of being nongeneric. Online Appendix F provides some examples based on the off-the-shelf approaches to obtaining a power law in economics (Gabaix, 2009), albeit a detailed analysis of possible microfoundations for the CD task technology is outside of the scope of this paper.

Second, as shown is Proposition 3, the underlying measure is infinite ($\mu(\mathcal{Q}) = \infty$), implying that the mass of tasks necessary to produce a unit of output is infinite. What makes the measure infinite is that the complexity space is extended toward $q = 0$ in (16), which is where the measure increases toward infinity.[24] Given the functional form for $k(q)$ in (16), this means

[24] Note that, for any $\varepsilon > 0$, we have $\int_{q_0}^{q_0+\varepsilon} q^{-2} \to_{q_0 \to 0} +\infty$.

that there is an exploding mass of tasks around $q = 0$, but since these tasks are trivial to be completed using capital (since $k(q) \to_{q \to 0} 0$), production is feasible. The key implication of this property is that capital becomes essential for production. Labor is also essential because of $k(q) \to_{q \to \infty} \infty$.

This second property is potentially problematic and undesirable because it implies that density g cannot be normalized to a probability density function (pdf, hereafter); that is T^{CD} does not have a probabilistic representation. As stressed in the beginning, a probabilistic representation of technology is useful for accounting for the firm- and sectoral-level heterogeneity in labor shares observed in the data while preserving the aggregation properties exhibited by the technology under our representative firm framework. Specifically, had the measure been finite, we could equivalently recast our model economy as comprising a finite mass of heterogeneous firms that draw a finite number of tasks from a common technology T each, and yet the aggregate production function would be the same as in our representative firm setup because all firms would be drawing tasks from the same pdf g associated with task technology T (assuming the law of large numbers holds on a continuum).[25] An extension along these lines to account for firm-level heterogeneity is not possible when the measure is infinite because it would require an infinite mass of firms. A similar observation has been one of the criticisms of the existing microfoundations of the CD function, and at least up to this point, it applies to our microfoundation as well.[26]

As we show in Online Appendix C, fortunately there is a straightforward remedy to this issue. The idea is to relax the strict requirement for an exactly balanced growth path, given that the data is measured with an error and that the Kaldor facts are merely statistical facts based on a finite sample of data. If so, an approximate balanced growth path suffices to account for these facts in a statistical sense, and this requirement is satisfied by the following domain-truncated

[25] As an example of an endogenous theory of firm heterogeneity, consider a setup along the lines of Atkeson and Kehoe (2007): Let $\{T_t\}_{t=0}^{\infty}$ be the sequence of technology implying balanced growth in our representative firm economy. Suppose that a subset of households called entrepreneurs has the technology to draw $N \in \{1, 2, 3...\}$ tasks at some fixed cost from the current $T_t = \{..., g_t\}$ technology according to pdf g_t and can establish firms. Each firm uses capital and labor to produce output by completing its own tasks in a cost minimizing way, and output is y_i^v for a given firm i, where $0 < v < 1$ is the span of control and y_i is the number of times all tasks are completed by firm i. Suppose that the firm distribution evolves, with old firms being retired as new firms draw technology from the improving technology frontier T_t. In such a setup, the aggregation properties exhibited by T_t would largely carry over to the heterogeneous firm setup after accounting for the effect of a limited span of control and positive profits in equilibrium.

[26] See footnote 8.

version of the CD task technology:

$$\left\{k\left(q\right)=Z^{-1}\left(q+q_0\right)^{\frac{1}{\alpha}}\frac{1-\alpha}{\alpha},g\left(q\right)=A^{-1}\left(q+q_0\right)^{-2}\right\},\tag{20}$$

where $q_0 > 0$ is the approximating parameter given $q_0 = 0$ yields (16). The underlying density here is the Pareto probability density, and hence this technology does have probabilistic representation.

Online Appendix C shows that the above domain-truncated technology exhibits approximately the same balanced growth path for sufficiently low q_0 as the canonical CD technology and it further converges to the CD model's balanced growth path over time. Factor shares are approximately constant and they converge to constant shares over time. The production function is approximately CD along the balanced growth path and it converges uniformly to CD production function on any bounded domain. The truncation of the the task domain effectively implied by q_0 is interesting in its own right because it implies that capital is not used in production until a threshold productivity level is reached (in terms of Z or A). As a result, this model provides additional task-based microfoundations for the model of industrial revolution by Hansen and Prescott (2002), or under an appropriate extension for models of poverty traps driven by learning-by-doing externality associated with using capital à la Romer (1986).

3 Modeling IT revolution in automation

Next, we use our theory to propose a concrete model of how the emergence of IT-based automation adopted by profit-maximizing firms can endogenously become labor-share displacing along the balanced growth path. The key feature of this model is that firms can use IT to "compress" the task load required to produce a machine (capital) specific to a given task at the expense of completing a fixed measure of some other tasks, with the degree of compression being optimally chosen by the firm. The idea is that the fixed set of tasks is associated with adding a computer chip and/or lines of computer code to obtain "smart" machines that optimize the use of hardware. Intuitively, IT is labor-share displacing according to this model because it constitutes a "complexity-bias" form of technical progress. As a result, the diffusion argument does not apply, and IT correlates technical change across tasks and permanently reduces the labor share.

3.1 Generalized setup with endogenous complexity

Suppose capital is task-specific and must be produced by completing tasks. In particular, define a machine of type q as a lasting embodiment of tasks, which, if completed once, can be used repeatedly to complete task q through the use of this machine until a Poisson event with arrival rate δ ends its useful lifetime. Suppose, the technology to produce a machine of type q involves complexity space $\mathcal{Q} = \mathbb{R}_+$ and a q-specific density $\tilde{g}_q(\hat{q})$ of tasks that must be completed at each complexity level $\hat{q}$ by either capital or labor. As a result, technology is now only described by the density function; that is, by $\tilde{T}_q = \{\tilde{g}_q\}$ in the capital goods sector(s) producing machines and by $T = \{g\}$ in the goods sector producing consumption goods.

The schedule $k(q)$ is endogenous to technology and it pertains to the (real) purchase price of a machine of type q. The production of a task q-specific machine is assumed to happen in an instant of time and the firm that produces machines is assumed to make zero profits. Accordingly, in equilibrium, the $k(q)$ schedule satisfies the fixed point:

$$Pk(q) := \min_{\{\mathcal{Q}_k, \mathcal{Q}_l\} \in \mathscr{P}} \left\{ r \int\limits_{\hat{q} \in \mathcal{Q}_k} k(\hat{q})\, \tilde{g}_q(\hat{q})\, dv + w \int\limits_{\hat{q} \in \mathcal{Q}_l} \tilde{g}_q(\hat{q})\, dv \right\}. \tag{21}$$

As in the baseline model, the production cost on the right comprises the cost of inputs used to complete tasks associated with producing one machine of type q. The machine-producing firm seeks a measurable partition $\mathscr{P}$ that minimizes costs, which includes the collection of all measurable partitions of the complexity space $\mathcal{Q}$. By definition, tasks assigned to capital are in set $\mathcal{Q}_k$. The user cost of a machine is $rk(q)$ to maintain consistency with the baseline model, and in Online Appendix D we explicitly derive it to show that here it is given by $r_t = P_t(1 + \rho - (1 - \delta)\gamma_{r,t})$, where $\gamma_{r,t}$ is the growth rate of $P_t k_t(q)$ at t. Tasks assigned to labor are in set $\mathcal{Q}_l$ and are treated analogously.

We next specialize this setup to focus attention on the balanced growth path, which we achieve by imposing the following assumption:

Assumption 2. $\tilde{g}_q(\hat{q}) = \lambda(q)\, \tilde{g}(\hat{q})$, *where* $\tilde{g}(\hat{q}) = A^{-1}\hat{q}^{-2}$ *and* $\lambda(q) = Z^{-1} q^{\frac{1}{\alpha}}$.

This assumption implies that the density of tasks needed to produce a machine of type q involves some *base density* $\tilde{g}(\hat{q}) = A^{-1}\hat{q}^{-2}$—which we already know yields CD production function (see Example 1)—and a *task load* function $\lambda(q)$ that scales it. Intuitively, tasks that are more complex involves the complexity mix of tasks on average but they require a larger overall measure of tasks to be completed.

We next show that the extended setup endogenously yields the capital requirement function associated with the CD task technology in (16); that is, $k(q) = Z^{-1}q^{\frac{1}{\alpha}}$. Accordingly, the production function is CD in the capital-producing sector(s) and hence also in the consumption sector as long as $g(q) = A^{-1}q^{-2}$ (as in Example 1). We summarize this result in the proposition below. The key additional result to note is that the functional form for λ_q is "inherited" by the $k(q)$ schedule as labeled.[27] We use this property in the next section.

Proposition 4. *The production function in the capital q-producing sector is Cobb–Douglas:*

$$Y = Zq^{-\frac{1}{\alpha}}A\left(Z\left(\frac{\tilde{c}(w,r)}{P}\right)^{-1}K\right)^{1-\alpha}L^{1-\alpha},$$

and the implied capital requirement function inherits the shape of task load schedule λ_q and takes the form:

$$k(q) = \underbrace{Z^{-1}q^{\frac{1}{\alpha}}}_{=\lambda(q)}\frac{\tilde{c}(w,r)}{P}, \tag{22}$$

where $\tilde{c}(w,r)$ is the unit cost of production in the capital producing sector associated with the base technology $\{\tilde{g}\}$. If, in addition, $g(q) = A^{-1}q^{-2}$, then the production function in the goods sector is also Cobb–Douglas of the form: $Y = A(ZK)^{\alpha}L^{1-\alpha}$.

3.2 Labor-share displacing effect of IT-powered automation

We are now ready to lay out our model of IT. The idea is that IT is associated with a discrete discovery of a universal technology with certain properties that becomes available to the economy at a point in time and was not available before. Its arrival will permanently change the labor share in the economy that will converge to the new balanced growth path. As mentioned, IT is a technology that allows capital-producing firms to "compress" the task load required to produce a machine (capital) at the expense of completing a fixed measure of some other tasks. The

[27] While we do not explore it, an interesting feature of the extended model is the endogenous propagation of technical progress via the fixed point involved in the production of capital goods. This is apparent from the functional forms, which involve a unit cost for completing tasks in the capital-producing sector associated with the base density $\tilde{g}^0$.

costs associated with operating this technology are assumed to be borne each time a machine is produced. The formal definition is as follows:

Definition 3. A *breakthrough IT automation technology* comprises i) a task technology $T^{IT} = \{g^{IT}\}$ and ii) an associated strictly decreasing compression function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$, such that T^{IT} used $n \geq 0$ times "compresses" the task load in the production of machines of type $q \in \mathcal{Q}$ so that density becomes $\tilde{g}_{q,n}(\hat{q}) = \lambda_q \kappa(n) \tilde{g}(\hat{q})$. (Units are sufficiently small to justify $n \in \mathbb{R}_+$.)

Assumption 3. $\kappa(n) = \kappa_0 \beta^{-1} n^{-\beta}$, *where* $0 < \beta < \alpha^{-1} - 1$ *and* $\kappa_0 > 0$ *are scalars.*

Assumption 3 specializes the functional forms to ensure that in the long run the arrival of this technology is consistent with balanced growth. The assumption states that a capital-producing firm can reduce the task load λ_q by $100 * \beta$ percent at the expense of completing a fixed measure of tasks associated with IT technology T^{IT} once. The crucial assumption here is the scalability of this technology; that is, the fact that the firm can repeat the process n times to reduce the task load by $100 * \beta$ percent n times.

In what follows, we examine its effects of this technology introduced into the setup underlying Proposition 4. We assume the initial capital cost schedule is given by $k_0(q) = q^{\frac{1}{\alpha}}$ (with $Z = 1$ for simplicity).

Results

Based on the results obtained in Proposition 4, the breakthrough technology implies that the new capital price schedule is:

$$k(q) = \min\left\{q^{\frac{1}{\alpha}}, \min_{n \geq 0} \kappa(n) q^{\frac{1}{\alpha}} + bn\right\}, \tag{23}$$

given initial schedule $k_0(q) = q^{\frac{1}{\alpha}}$, and given the observation that the reduction in task load λ_q scales down proportionally the cost of producing a machine of type q—as noted under equation (22) in Proposition 4. The lowest cost of completing tasks associated with IT is denoted $b > 0$, which we do not need to specify explicitly because, from an atomless firm's point of view, this is just a constant. Of course, in equilibrium, b is linked to factor prices, resulting in a fixed point on the economy-wide level as the technology is applied.[28]

The inner minimization problem in (23) implies that the optimum occurs at $n^* = \left(\frac{C}{b} q^{\frac{1}{\alpha}}\right)^{\frac{1}{1+\beta}}$, which gives

$$k(q) = \min\left\{q^{\frac{1}{\alpha}}, \frac{b}{\beta}\left(\frac{\kappa_0}{b} q^{\frac{1}{\alpha}}\right)^{\frac{1}{1+\beta}}\left(\left(\frac{\kappa_0}{b}\right)^{\frac{1}{1+\beta}} + 1\right)\right\}. \tag{24}$$

Accordingly, the transformed schedule is of the form $k(q) \propto q^{\frac{1}{\alpha}\frac{1}{1+\beta}}$, after we remove the constants that are not relevant to the determination of the labor share. It is clear from Example 1 that had $k(q) \propto q^{\frac{1}{\alpha}\frac{1}{1+\beta}}$ been applied globally to production of goods, we would have obtained a CD aggregate production function featuring a labor share given by $LS_1 = LS_0 - \beta\alpha$ instead of the initial $LS_0 = 1 - \alpha$. However, the min operator implies that the breakthrough technology may not be adopted globally, resulting in a partial transformation of the k function to this new functional form.

As we show in the appendix, the benefit from applying the breakthrough technology is strictly increasing in q, and hence the above problem implies a cutoff value $q_{\min} > 0$ such that the technology is applied only into the production of machines of type q above that cutoff. The resulting schedule is thus of the form:

$$k(q) = \begin{cases} q^{\frac{1}{\alpha}} & q \le q_{\min} \\ C q^{\frac{1}{\alpha}\frac{1}{1+\beta}} & q \ge q_{\min} \end{cases},$$

where C is a scalar satisfying $q_{\min}^{\frac{1}{\alpha}} = C q_{\min}^{\frac{1}{\alpha}\frac{1}{1+\beta}}$, and the lower b is, the lower $q_{\min}$ is. The fact that the new capital requirement does not apply to the lower portion of the domain introduces an isoquant error with respect to the canonical CD production function's isoquant. However, the impact of this feature—identical to that of the truncation of the domain introduced by $q_0 > 0$ in (20) and analyzed in Online Appendix C—vanishes with the growth arising through automation. As a result, the decline in the labor share in this economy, while smaller initially, in the limit converges to $LS_1 = LS_0 - \beta\alpha$, where $LS_0 = 1 - \alpha$ is the initial labor share, and the economy will return to a new balanced growth path. We summarize this results in the proposition below.

[28]The effect of solving for this fixed point, as we will see, implies that the technology is applied more globally across on complexity space $\mathcal{Q}$, which only reinforces the results because the outer minimum has less bite.

Proposition 5. *Suppose that the labor share is $LS_0 = 1 - \alpha$. The post-breakthrough labor share converges to $LS_1 = LS_0 - \alpha\beta$ as the economy further automates so that $q_{min}/q^* \to 0$.*

We have now identified the properties that a transformative IT technology operating on the task space must exhibit to be labor-share displacing. Our model is stylized, but it is a step forward toward establishing the correspondence between the technologies that have powered the current wave of automation and what is qualitatively needed for those technologies to adversely impact the labor share according to our framework. We now conclude by briefly discussing the key lessons we draw from this analysis in light of the anecdotal evidence regarding the transformative role IT plays in the current wave of automation. A detailed discussion of actual technologies is beyond the scope of this paper—and can be found in the monographs by Brynjolfsson and McAfee (2014), Ford (2009) and Frey (2020). We use our reading of the conclusions reached by these authors to anchor the discussion.

Discussion: Is modern IT-powered automation labor-share displacing?

A breakthrough technology lowers the labor share in our model because it exhibits three key qualitative properties: 1) universal applicability, 2) task measure compression, and 3) scalability. The first property means that the breakthrough technology can be applied to most types of capital-producing tasks, implying a global impact on production. The second property means that IT "compresses" the density of tasks in the production of capital goods, implying a larger payoff from its application in the case of more complex tasks. The third feature allows for a scalable application of this technology to achieve a larger effect when the payoff is larger. All these features are critical for technology to be labor-share displacing, but the exact functional forms are not.

Through the lens of our theory, then, the question of whether the modern IT-powered wave of automation is labor share-displacing comes down to the question of whether the kind of enabling IT technologies that power the current wave of automation exhibit these properties in some shape or form. While this is a difficult question to answer conclusively, there are some parallels between our model and the data on an anecdotal level.

Take, for example, the line of reasoning in Brynjolfsson and McAfee (2014), who coined the phrase "the second machine age" to draw a stark distinction of this era and preceding "first machine age" that bunches together what economic historians refer to as the first and the second industrial revolution.

The key qualitatively novel aspect that they emphasize is that the automation enabled by advanced IT technologies and digitization allows capital to permeate tasks that are cognitively intensive—while implying that such tasks were prohibitively costly to automate in the past or required costly hardware workarounds (e.g., an automated production line). According to Brynjolfsson and McAfee (2014), capital in the past largely delivered mechanical power that served as a substitute for human or animal power, and its application without the assistance of humans was limited to highly structured environments. Automation was possible, but it was costly because it required a conversion of cognitively demanding unstructured processes into highly structured noncognitive processes.

This line of reasoning suggests that, first, the technology involved is in principle universal because production broadly involves cognitive reasoning. The question is more about the scope and the maturity of the existing technologies, since current technologies only partially overcome this hurdle (e.g., AI). Second, if cognition was such an obstacle to automation, the technologies that overcome it targets the most complex tasks for the capital of the past. Finally, third, to the extent that cognitively intensive tasks are disproportionately more costly to automate, a disproportionate payoff from overcoming this hurdle for a fixed amount of IT capital is expected, which is conducive to the presence of compression and scalability. Whether all these forces work as in our model is unclear, but the broad direction of technical change appears to exhibit some parallels vis-à-vis what is critical for automation to be labor-share displacing.

4 Conclusions

We characterized how automation affects the division of income between capital and labor in the context of long-run growth. In the process, we developed an analytically tractable task-based microfoundation for the Cobb–Douglas production function. We proposed a stylized model of IT-driven technological breakthrough and identified the key conditions under which the IT-powered automation may be labor-share displacing. While we have argued that technical change of a more diffused form can be favorable to labor, we also concluded that the modern wave of IT-powered automation may turn out to be labor-share displacing because it involves a universal technology that directly targets complexity; that is, it can be interpreted as "complexity–biased" technical change through the lens of our theory.

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Appendix: Omitted proofs

Proof of Lemma 1 **Part I:** We first show that constraints in (4) are satisfied for finite inputs when a cutoff rule is used; in particular, we need to show that i) the constant function $f(q) = 1$ is μ-integrable, or equivalently $g(q)$ is Lebesgue integrable on interval $[q^*, \infty)$, and that ii) $k(q)$ is μ-integrable, or equivalently $k(q)\, g(q)$ is Lebesgue integrable on interval $[0, q^*]$, where q^* satisfies the requirement of the lemma. **Step 1:** Assume $q^* < \infty$ and $k(q^*) > 0$. To establish property (i), define $\mathcal{S} = [q^*, \infty)$. By contradiction, suppose $g(q)$ is not Lebesgue integrable on $\mathcal{S}$ (i.e., $\int_{\mathcal{S}} d\mu = \int_{\mathcal{S}} g(q)\, dv = +\infty$, since measurability is assumed and g is non-negative). By Assumption 1, there is a measurable partition of $\mathcal{S}$ to two disjoint subsets $\mathcal{S}_l = \mathcal{Q}_l \cap \mathcal{S}, \mathcal{S}_k = \mathcal{Q}_l \cap \mathcal{S}$ such that $\int_{\mathcal{S}_l} d\mu < \infty$ and $\int_{\mathcal{S}_k} k(q)\, d\mu < \infty$, where $\{\mathcal{Q}_l, \mathcal{Q}_k\}$ is the partition implied by that assumption. Since $k(q)$ is an increasing function, we must have $\infty > \int_{\mathcal{S}_k} k(q)\, d\mu \geq \int_{\mathcal{S}_k} k(q^*)\, d\mu$, which leads to a contradiction because

$$\infty > \int\limits_{\mathcal{S}_l} 1 d\mu + \int\limits_{\mathcal{S}_k} k(q)\, d\mu \geq \int\limits_{\mathcal{S}_l} 1 d\mu + k(q^*) \int\limits_{\mathcal{S}_k} d\mu = (1 + k(q^*)) \int\limits_{\mathcal{S}} 1 d\mu = +\infty. \tag{25}$$

To establish property (ii), note that $0 \leq k(q) \leq \frac{w}{r} < \infty$ for all $q \leq q^*$, implying $\frac{r}{w} k(q) < 1$. Accordingly, we obtain $\int_{[0,q^*]} k(q)\, d\mu < \infty$ from the following chain of evaluations:

$$\begin{aligned}
\infty > \int\limits_{\mathcal{Q}_l} 1 d\mu + \int\limits_{\mathcal{Q}_k} k(q)\, d\mu &\geq \int\limits_{\mathcal{Q}_l} \frac{r}{w} k(q)\, d\mu + \int\limits_{\mathcal{Q}_k} k(q)\, d\mu \\
= \left(\frac{r}{w} + 1\right) \int\limits_{\mathcal{Q}} k(q)\, d\mu &\geq \left(\frac{r}{w} + 1\right) \int\limits_{[0,q^*]} k(q)\, d\mu.
\end{aligned}$$

(For Step 2 below, note that the proof of property (i) does not depend on q^* satisfying the requirement of the cutoff from the statement of the lemma in text, and the proof of property (ii) does not depend on the assumption that $k(q^*) > 0$.) **Step 2:** Consider now the degenerate cases: a) $k(q^*) = 0$ ($0 \leq q^* < \infty$) or b) $q^* = +\infty$ (a and b is impossible by the assumption made in text that k becomes strictly positive for a sufficiently large q). **Case a**: By definition of the cutoff and monotonicity of $k(q)$, we must have $k(q) \geq \frac{w}{r} > 0$, for all $q > q^*$, and $k(q) = 0$ for all $q \leq q^*$ (the strictly inequality follows here from the hypothesis that $k(q^*) = 0$). Accordingly, we have established property (i), since $\int_{[0,q^*]} k(q)\, d\mu = \int_{[0,q^*]} 0 d\mu = 0$. Recall that, as noted, the argument used in Step 1 (proof of property ii) did not require $k(q^*) > 0$, and so property (ii) has already been proven. **Case b**: $q^* = \infty$ implies $k(q) \leq \frac{w}{r}$, for all $q \in \mathcal{Q}$. Accordingly, by Assumption 1 and the fact that $\frac{r}{w} k(q) \leq 1$ globally, property (i) follows from the evaluation:

$$\infty > \int\limits_{\mathcal{Q}_l} 1 d\mu + \int\limits_{\mathcal{Q}_k} k(q)\, d\mu \geq \frac{r}{w} \int\limits_{\mathcal{Q}_l} k(q)\, d\mu + \int\limits_{\mathcal{Q}_k} k(q)\, d\mu = \left(\frac{r}{w} + 1\right) \int\limits_{\mathcal{Q}} k(q)\, d\mu.$$

To see that $\lim_{q^* \to \infty} \int_{q^*}^{\infty} 1\mu = 0$, we apply the argument used in Step 1 (proof of property i) to show that $\int_{q^{**}}^{\infty} 1\mu$ exists (is finite) for sufficiently large q^{**} such that $k(q^*) > 0$ (the existence of such a sufficiently large and finite q^{**} is ensured by the fact that $k(q)$ is strictly positive on at least part of the domain and, as noted, Step 1 (proof property i) did not rely on q^* being the cutoff). Since for any Lebesgue integral we have $\lim_{q^* \to \infty} \int_{q^*}^{\infty} 1\mu = 0$, both constraints in 4 are well defined.[29] This finishes the first part. **Part II:** By contradiction, suppose now that there exists a task partition $\mathcal{Q}_k = \mathcal{E}$ of a positive measure under μ that solves the minimization problem and that is different from that implied by the cutoff rule in the lemma (on a set of positive measure). If so, reassigning production of tasks in $\mathcal{A} = \mathcal{E} \cap \{q : q > q^*\}$ from capital to labor must reduce the cost because $rk(q) > w$ on that set by definition of the cutoff rule, since we are minimizing $rK + wL$, and analogously for $\mathcal{A}^c$ when switched from labor to capital. At least one of these sets must be of positive measure by assumption, contradicting cost minimization. Q.E.D.

[29]The proof follows the fact that the tail sum of any convergent series converges to zero, that is, if $\sum_{i=1}^{\infty} a_i$ converges, then $t_n = \sum_{i=n}^{\infty} a_i \to_{n \to \infty} 0$, which itself is a corollary from Cauchy's criterion of convergence for series. Specifically, define $a_i = \int_{q^*+i-1}^{q^*+i} 1 d\mu$, note that $\sum_{i=1}^{\infty} a_i = \int_{q^*}^{\infty} 1\mu < \infty$ by Theorem 5.24 from Wheeden and Zygmund (1977) and the hypothesis, and now apply the result for series.

Proof of Lemma 2 Consider the definition of the production function in (5) with equality:

$$Y(K,L) := \sup\left\{Y : \exists_{q^*\in\mathcal{Q}} \text{ s.t. } K = Y\int_0^{q^*} k(q)\,d\mu, L = Y\int_{q^*}^{\infty} 1 d\mu\right\}, \tag{26}$$

We split the proof to two steps: Step 1) the solution (Y,q^*) to the above equations exists. Step 2) the solution from Step 1 attains the supremum under the original formulation in (5). **Step 1**: Note that the constraint in (26) implies that q^* satisfies

$$\frac{L}{K} = \frac{\int_{q^*}^{\infty} 1 d\mu}{\int_0^{q^*} k(q)\,d\mu}. \tag{27}$$

The integral in the numerator is finite whenever the integral in the denominator is nonzero. This follows from the proof of Lemma 1 where we have established exactly this property (Part I, Step 1). The key here is that when the denominator (or $K > 0$) is positive, then $k(q^*) > 0$, which in turn implies the existence (finiteness) of the integral in the numerator by the argument used in the proof of Lemma 1 (Part I, Step 1, proof of property i). Note the following properties of the expression on the right-hand side of (27) as a function of q^*: i) the numerator can be made arbitrarily small as $q^* \to 0$, and since the numerator is increasing in q^*, the expression goes to ∞ as $q^* \to 0$; ii) the numerator goes to 0 when $q^* \to \infty$, and since the denominator is positive and increasing in q^*, the expression goes to 0 as $q^* \to \infty$ (see explicit proof in footnote 29); finally, iii) the expression is continuous with respect to q^* and strictly decreasing, hence bijective on $\mathbb{R}_+$.[30] Accordingly, there exists a unique $0 < q^* < +\infty$ that satisfies the two constraints (for any finite L/K). Furthermore, the supremum is attained within this set because without a loss we can restrict attention to a compact domain of $\left(\hat{Y}, q^*\right)$ while maximizing a continuous function $f\left(\hat{Y}\right) = \hat{Y}$ on the set defined by (26). Accordingly, maximum exists by the Weierstrass extreme value theorem. **Step 2:** We now turn to the question of whether this solution attains the supremum under the original definition of the production function in (5). For now assume $K > 0$, and we will return to this. Suppose, by the way of contradiction that there exists $\hat{Y}' > \hat{Y}$, $q^{*\prime} > 0$ such that $K > \hat{Y}'\int_0^{q^{*\prime}} k(q)\,g(q)\,dv, L \geq \hat{Y}'\int_{q^{*\prime}}^{\infty} g(v)\,dv$ (the

[30]This follows from the fact that Lebesgue integral is continuous function of the bounds of integration. This is a known corollary from dominated converge theorem. Lacking a textbook reference, we prove it in the Online Appendix E.

case $K = \hat{Y}' \int_0^{q^{*\prime}} k(q)\, g(q)\, dv, L > \hat{Y}' \int_{q^{*\prime}}^{\infty} g(v)\, dv$ will follow by analogy and it is omitted). Cases like this imply that the supremum of the original problem must exceed the one implied by 26 and we eliminate such a possibility by showing this leads to a contradiction. To that end, note that the integrals exist at $q^{*\prime}$ by the hypothesis (the inequalities as stated). By the continuity of Lebesgue integrals (see comment in footnote 30), we can pick $\Delta q^{*\prime} > 0$ such that $K > \hat{Y} \int_0^{q^{*\prime}+\Delta q^{*\prime}} k(q)\, g(q)\, dv$, which also implies that there exists $\Delta \hat{Y} > 0$ such that $K = \left(\hat{Y}' + \Delta \hat{Y}'\right) \int_0^{q^{*\prime}+\Delta q^{*\prime}} k(q)\, g(q)\, dv$ (by continuity of the expression on the right). The integral on the extended interval up to $q^{*\prime} + \Delta q^{*\prime}$ exists for the following reason. Let $\bar{k} := \sup_{[q^{*\prime}, q^{*\prime}+\Delta q^{*\prime}] \subset \mathcal{Q}} k(q)$, which must be a finite number. If this is not the case, we necessarily have $k(q^{**}) = +\infty$ for any $q^{**} > q^{*\prime} + \Delta q^{*\prime}$, simply because $k(q)$ is monotone (having $k(q^{**}) = +\infty$ on part of the domain contradicts that k is a real-valued function (hence finite-valued) and defined everywhere on $\mathcal{Q}$). Using this fact, the following chain of evaluations show that the integral in question exists as long as $\int_{q^{*\prime}}^{\infty} g(v)\, dv$ exists, which has been ensured by the hypothesis:

$$\infty > \bar{k} \int_{q^{*\prime}}^{\infty} g(q)\, dq > \bar{k} \int_{q^{*\prime}}^{q^{*\prime}+\Delta q^{*\prime}} g(q)\, dq > \int_{q^{*\prime}}^{q^{*\prime}+\Delta q^{*\prime}} k(q)\, g(q)\, dq.$$

Returning to the main argument, the fact that g has full support implies $L > \hat{Y}' \int_{q^{*\prime}+\Delta q^{*\prime}}^{\infty} 1 g(v)\, dv$ by the continuity of the Lebesgue integral.[31] But, if so, there exists $\hat{Y}'' = \hat{Y}' + \Delta \hat{Y}''$, for some $0 < \Delta \hat{Y}'' < \Delta \hat{Y}'$, such that $K \geq \hat{Y}'' \int_0^{q^{*\prime}+\Delta q^{*\prime}} k(q)\, g(q)\, dv$ and we maintain $L = \hat{Y}'' \int_{q^{*\prime}+\Delta q^{*\prime}}^{\infty} g(v)\, dv$, which is a contradiction of the fact that $\hat{Y}' > \hat{Y}$. $\hat{Y}' = \infty$ is not feasible by assumption of k being strictly positive on part of the support. The remaining case is eliminated analogously by considering "$-\Delta q^{*\prime}$" and we omit the details. If $K = 0$, there is not much to prove because $q^* = q_0$. Q.E.D.

Proof of Lemma 3 Part I shows existence and Part II derives the formulas. **Part I:** The proof builds on the proof of Lemma 2. We have established in that lemma that the production function can be obtained from 26 and that a unique q^* exist that satisfies (27). By the second constraint

[31] See footnote 30.

then, we know that $Y(K,L), q^*$ satisfy

$$L = Y(K,L)\,\mu([q^*,\infty)), \tag{28}$$

which gives

$$q^*(Y,L) = S^{-1}\left(\frac{L}{Y}\right), \tag{29}$$

where $S(q) := \mu([q,\infty))$ is the survival function. The survival function under the assumptions made in text, by previous lemmas, is a well defined, positively-valued, continuous, strictly decreasing (because g has full support), and hence invertible and differentiable almost everywhere with a strictly negative derivative.[32] $S^{-1}\left(\frac{L}{Y}\right)$ exists and is differentiable a.e., since for functions of a single variable we have $[f^{-1}]'(x) = \frac{1}{f'(f^{-1}(x))}$, which is well defined as long as f' is nonzero (which it is). We have now shown that the derivative of $q^*(Y,L)$ in (29) is well defined (a.e.). The production function $Y(K,L)$ can be recovered from the capital usage equation in 2, which implies the identify:

$$f(Y(K,L),K) := Y(K,L)\int_0^{q^*(Y(K,L),L)} k(q)\,d\mu - K \equiv 0.$$

The implicit function theorem now ensures that at the points of differentiability of $q^*(Y,L)$ the partial derivative $Y_K(K,L)$ is well defined as long as the partial derivative $f_Y(Y,K) := \frac{\partial f(Y,K)}{\partial Y}$ is non-vanishing (nonzero), and both f_Y, f_K are well defined, which is implied by the above functional form. The existence of the partial derivative with respect to L can be shown analogously. **Part II:** Here we derive the formulas for output elasticity with respect to capital. The formula for labor follows by Euler's law under constant returns to scale and we focus on the formula for capital. Consider fixed $K > 0$ and fixed $L > 0$. By the proof of the previous lemma we can assume this point is associated with output $Y > 0$ by (26) and some cutoff $q^* > 0$ that solve (27). Consider now an infinitesimal increment $dK > 0$ that adds to the capital stock (we consider a positive value to fix idea but the reasoning will not depend on that). Given the change in the optimal cutoff $dq^* > 0$ in response to the increase in capital by $dK > 0$, we can calculate $dY > 0$ from the labor input equation ($L = Y\int_{q^*}^{\infty} 1 d\mu$), which

[32]See Theorem 7.21 (p. 111) in Wheeden and Zygmund (1977).

implies $dL = (Y + dY) \int_{q^*+dq^*}^{\infty} 1d\mu - Y \int_{q^*}^{\infty} 1d\mu = 0$. By continuity of the Lebesgue integral,[33] the increase in labor is infinitesimal, and also $dq^* > 0$. Accordingly, we have

$$Y = dY \frac{\int_{q^*+dq^*}^{\infty} 1d\mu}{\int_{q^*}^{q^*+dq^*} 1d\mu}. \tag{30}$$

By (26) ($K = Y \int_0^{q^*} k(q)\, d\mu$), $dK = (Y + dY) \int_0^{q^*+dq^*} kd\mu - Y \int_0^{q^*} kd\mu$, which after plugging in for Y from (26) and dividing both sides by dY gives

$$\frac{dK}{dY} = \int_0^{q^*+dq^*} kd\mu + \frac{\int_{q^*+dq^*}^{\infty} 1d\mu}{\int_{q^*}^{q^*+dq^*} 1d\mu} \int_{q^*}^{q^*+dq^*} kd\mu.$$

(Recall that $k(q^*) > 0$, implying $dY > 0$, which is ensured by the fact that we started with $K > 0$ and $k(q)$ is an increasing function.) By the analog of the intermediate value theorem for Lebesgue integrals (e.g., Wheeden and Zygmund (1977), Corollary 5.31, p. 75), we can thus pick a number $k(q^*) \leq \hat{k} \leq k(q^* + dq)$ such that $\int_{q^*}^{q^*+dq^*} kd\mu = \hat{k} \int_{q^*}^{q^*+dq^*} 1d\mu$, implying the above equation simplifies to

$$\int_0^{q^*+dq^*} kd\mu + \hat{k} \int_{q^*+dq^*}^{\infty} 1d\mu = \frac{dK}{dY}.$$

By (5), observe that $\int_0^{q^*} k(q)\, d\mu = \frac{K}{Y}$, $\int_{q^*}^{\infty} 1d\mu = \frac{L}{Y}$,and so the above equation can be rewritten as $\frac{K+dK}{Y+dY} + \hat{k}\frac{L}{Y+dY} = \frac{dK}{dY}$. Given dK, dY are infinitesimal, we can ignore them in the first two terms on the left-hand side and use $\frac{K}{Y}$ in place of $\frac{K+dK}{Y+dY}$. Multiplying both sides by $\frac{Y}{L}$, and using the fact that $k(q^*) \leq \hat{k} \leq k(q^* + dq)$ a.e. implies $\hat{k} \to k(q^*)$ as $dq^* \to 0$, which gives $\frac{dK}{dY}\frac{Y}{K} = 1 + k(q^*)\frac{L}{K}$ a.e. This finishes the proof for α. The elasticity of output with respect labor can be recovered from the Euler's law by the fact that the production function is constant returns to scale, implying $\frac{\partial Y}{\partial K} K/Y + \frac{\partial Y}{\partial L} L/Y = 1$. We have already shown it applies a.e. Q.E.D.

[33] See footnote 30.

Proof of Proposition 1 Given $\frac{L}{Y}\left(q_{\varepsilon}^{*}\right) := \int_{q_{\varepsilon}^{*}}^{\infty} g\left(q\right) dq$, Lebesgue differentiation theorem implies that

$$\frac{d\frac{L}{Y}\left(q_{\varepsilon}^{*}\right)}{d\varepsilon}|_{\varepsilon=0} = -g\left(q_{0}^{*}\right)\frac{dq_{\varepsilon}^{*}}{d\varepsilon}|_{\varepsilon=0} \text{ (a.e.)}$$

Accordingly, we derive the displacement effect from its definition in text:

$$DE := \frac{d\log w\frac{L}{Y}\left(q_{\varepsilon}^{*}\right)}{d\varepsilon}|_{\varepsilon=0} = \frac{\frac{d\left(w\frac{L}{Y}\left(q_{\varepsilon}^{*}\right)\right)}{dr}}{w\frac{L}{Y}\left(q_{\varepsilon}^{*}\right)}|_{\varepsilon=0} = -\frac{g\left(q_{0}^{*}\right)\frac{dq^{*}(\varepsilon)}{d\varepsilon}}{\int_{q^{*}}^{\infty} g\left(q\right)dq} = \frac{g\left(q_{0}^{*}\right)\frac{v\left(q_{0}^{*}\right)}{k'\left(q_{0}^{*}\right)}}{\int_{q^{*}}^{\infty} g\left(q\right)dq} = -h\left(q_{0}^{*}\right)\frac{v\left(q_{0}^{*}\right)}{k'\left(q_{0}^{*}\right)}, \tag{31}$$

and note here that $h\left(q\right) = -\frac{dS(q)}{dq}$. Consider now the equilibrium price change P_{ε} that, as noted in text, must satisfy the zero profit condition as an identity, i.e., $P_{\varepsilon} - \left(w\frac{L}{Y}\left(q_{\varepsilon}^{*}\right) + r\frac{K}{Y}\left(q_{\varepsilon}^{*}\right)\right) \equiv 0$. Differentiating this expression, we obtain

$$PE = -\frac{d\log P_{\varepsilon}}{d\varepsilon}|_{\varepsilon=0} = -\frac{dP_{\varepsilon}}{d\varepsilon}|_{\varepsilon=0}\frac{1}{P_{0}} = -\left(\frac{d\left(w\frac{L}{Y}\left(q_{\varepsilon}^{*}\right)\right)}{d\varepsilon}|_{\varepsilon=0} + \frac{d\left(r\frac{K}{Y}\left(q_{\varepsilon}^{*}\right)\right)}{d\varepsilon}|_{\varepsilon=0}\right)\frac{1}{P_{0}},$$

which gives

$$PE = -\left(-wg\left(q_{0}^{*}\right)\frac{dq_{\varepsilon}^{*}}{d\varepsilon}|_{\varepsilon=0} - r\int_{0}^{q_{0}^{*}} v\left(q\right)g\left(q\right)dq + rk\left(q_{0}^{*}\right)g\left(q_{0}^{*}\right)\frac{dq_{\varepsilon}^{*}}{d\varepsilon}|_{\varepsilon=0}\right)\frac{1}{P_{0}}.$$

Using optimality of q_{0}^{*}, we know $rk\left(q_{0}^{*}\right) = w$, and so the above expression simplifies to

$$-\frac{d\log P_{\varepsilon}}{d\varepsilon}|_{\varepsilon=0} = \frac{r}{P_{0}}\int_{0}^{q_{0}^{*}} v\left(q\right)g\left(q\right)dq.$$

Given (8), and the above equations for PE and DE, we calculate

$$\frac{d\log LS_{\varepsilon}}{d\varepsilon}|_{\varepsilon=0} = DE + PE = DE - \left(\frac{d\left(w\frac{L}{Y}\left(q_{\varepsilon}^{*}\right)\right)}{d\varepsilon} + \frac{d\left(r\frac{K}{Y}\left(q_{\varepsilon}^{*}\right)\right)}{d\varepsilon}\right)\frac{1}{P_{0}}.$$

Using (31), we have

$$\frac{d\log LS_{\varepsilon}}{d\varepsilon}|_{\varepsilon=0}=\frac{\frac{d\left(w\frac{L}{Y}(q_{\varepsilon}^{*})\right)}{d\varepsilon}}{w\frac{L}{Y}\left(q_{\varepsilon}^{*}\right)}|_{\varepsilon=0}-\underbrace{\frac{w\frac{L}{Y}\left(q_{\varepsilon}^{*}\right)}{P_0}|_{\varepsilon=0}}_{LS_0}\left(\frac{\frac{d\left(w\frac{L}{Y}(q_{\varepsilon}^{*})\right)}{d\varepsilon}}{w\frac{L}{Y}\left(q_{\varepsilon}^{*}\right)}|_{\varepsilon=0}+\frac{\frac{d\left(r\frac{K}{Y}(q_{\varepsilon}^{*})\right)}{d\varepsilon}}{w\frac{L}{Y}\left(q_{\varepsilon}^{*}\right)}|_{\varepsilon=0}\right).$$

By definition of LS_0 as labeled above and the differentiation of $\frac{d\left(r\frac{K}{Y}(q_{\varepsilon}^{*})\right)}{d\varepsilon}$, we obtain

$$\frac{d\log LS_{\varepsilon}}{d\varepsilon}|_{\varepsilon=0}=\left(1-LS_0\right)DE-LS_0\left(\frac{-\int_0^{q_0^*}v\left(q\right)g\left(q\right)dq+k\left(q_0^*\right)g\left(q_0^*\right)\frac{dq_{\varepsilon}^*}{d\varepsilon}|_{\varepsilon=0}}{\int_{q_0^*}^{\infty}g\left(q\right)dq}\frac{r}{w}\right)\text{ (a.e.)}.$$

Next, we plug in for $\frac{dq_{\varepsilon}^*}{d\varepsilon}|_{\varepsilon=0}$ from equation (9), use the fact that at the cutoff point we have $k_0\left(q_0^*\right)=\frac{w}{r}$, use equation (31), and use the definition of the hazard rate $h\left(q^*\right):=\frac{g(q^*)}{\int_{q^*}^{\infty}g(q)dq}$ to obtain

$$\frac{d\log LS_{\varepsilon}}{d\varepsilon}|_{\varepsilon=0}=h\left(q_0^*\right)LS_0\left(\frac{\int_0^{q_0^*}v\left(q\right)g\left(q\right)dq}{g\left(q_0^*\right)k\left(q_0^*\right)}\right)+\underbrace{-h\left(q_0^*\right)\frac{v\left(q_0^*\right)}{k'\left(q_0^*\right)}}_{=DE}\text{ (a.e.)}$$

The decline in the measure of the not yet automated tasks is obtained by evaluating $d\log S\left(q_{\varepsilon}^*\right)/d\varepsilon$ at $\varepsilon=0$ and using the expression for $\frac{dq_{\varepsilon}^*}{d\varepsilon}|_{\varepsilon=0}$ from text. Q.E.D.

Proof of Proposition 2 The first part of the proof is in text. The proof that $\gamma:=\gamma_K=\gamma_Y$ can be found in Jones and Scrimgeour (2008) and applies after adjusting for differences in notation. We finish the proof by establishing the following omitted steps: 1) $g_0\left(q\right)=g_0q^{-\zeta-1}$; 2) $\zeta=\frac{\gamma_Y+\gamma_g}{\gamma_{q^*}}$ $\theta=\frac{\gamma-\gamma_k}{\gamma_{q^*}}$, which guarantees $\theta<\zeta$, and also requires $\gamma-\gamma_k>0$; 3) $\gamma=\gamma_g-\alpha\gamma_k$, which we omit here because it follows from the next corollary and it is not essential for the existence of BGP. To show (1), note that

$$\frac{d\left(\int_{q^*(t)}^{\infty}g_0\left(q\right)e^{\gamma_g t}dq\right)/dt}{\left(\int_{q^*(t)}^{\infty}g_0\left(q\right)e^{\gamma_g t}dq\right)}=-\gamma,$$

since $\bar{L}$ is constant and $\bar{L}=Y_t\int_{q^*(t)}^{\infty}g_0\left(q\right)e^{\gamma_g t}dq$ (the equation follows from the definition of cutoff q^* in Lemma 2). Differentiating the above expression, dividing both sides by q^*, and using the fact that $\dot{q}^*/q^* = \gamma_{q^*}$ by the definition of BGP, we obtain

$$\frac{g_0(q^*)}{\int_{q^*}^{\infty} g_0(q)\, dq} = \frac{\gamma + \gamma_g}{\gamma_{q^*}} \frac{1}{q^*}. \tag{32}$$

This equation must apply to all q^* (starting from q_0), since q^* is assumed to be growing at a strictly positive rate. Accordingly, it is an identity. Define $f(q^*) := \int_{q^*}^{\infty} g_0(q)\, dq$ and note that f is almost everywhere differentiable with $f'(q^*) = -g_0(q)$ by the Lebesgue differentiation theorem.[34] Furthermore, note we can rewrite equation (32) as $\frac{f'(q^*)}{f(q^*)} \equiv -\frac{\gamma+\gamma_g}{\gamma_{q^*}}\frac{1}{q^*}$, which is an ordinary differential equation (ODE) and solves to $f(q^*) = Cq^{*-\frac{\gamma_Y+\gamma_g}{\gamma_{q^*}}}$ up to some constant C, which implies $g_0(q) = -f'(q^*) = C\left(\frac{\gamma+\gamma_g}{\gamma_{q^*}}\right) q^{*-\frac{\gamma+\gamma_g}{\gamma_{q^*}}-1}$. Accordingly, $g_0(q) = g_0 q^{*-\zeta-1}$, where $\zeta := \frac{\gamma+\gamma_g}{\gamma_{q^*}}$, and $g_0 = C\frac{\gamma+\gamma_g}{\gamma_{q^*}}$ is some positive constant. As for (2), we have already shown that $\zeta := \frac{\gamma+\gamma_g}{\gamma_{q^*}}$. The proof that $\theta := \frac{\gamma-\gamma_k}{\gamma_{q^*}}$ follows from the analogous reasoning applied to $K_t = Y_t \int_{q^*(t)}^{\infty} k_0(q)\, e^{\gamma_k t} g_0(q)\, e^{\gamma_g t} dq$, where we know from text that $k_0(q)$ is of the form $k_0(q) = k_0 q^{\theta}$, where $\theta > 0$, or else $T \notin \mathcal{T}^{ext}$. Accordingly, $\int_{q^*(t)}^{\infty} k_0(q)\, e^{\gamma_k t} g_0(q)\, e^{\gamma_g t} dq$ must be a constant, and θ can be obtained by solving

$$\frac{d\left(\int_{q^*(t)}^{\infty} k_0(q)\, e^{\gamma_k t} g_0(q)\, e^{\gamma_g t} dq\right)/dt}{\int_{q^*(t)}^{\infty} k_0(q)\, e^{\gamma_k t} g_0(q)\, e^{\gamma_g t} dq} = 0.$$

The approach to solve this differential equation is analogous to the previous step and we omit it. The solution gives $k_0(q)\, g_0(q) = C\left(\frac{\gamma_g+\gamma_k}{\gamma_{q^*}}\right) q^{*\frac{\gamma_g+\gamma_k}{\gamma_{q^*}}-1}$. Using the formula for $g_0(q)$, and the fact that $k_0(q) = k_0 q^{\theta}$, we obtain $\theta = \frac{\gamma-\gamma_k}{\gamma_{g^*}}$. Q.E.D.

Proof of Corollary 1 Using Lemma 2, we derive the production function from the conditions in the lemma, which are:

$$\frac{L}{Y} = g_0 \int_{q^*}^{\infty} q^{-\zeta-1} dq = g_0 \frac{1}{\zeta} q^{*-\zeta},\ \frac{K}{Y} = k_0 g_0 \int_{0}^{q^*} q^{\theta} q^{-\zeta-1} dq = \frac{k_0 g_0}{\theta-\zeta} q^{*\theta-\zeta}.$$

Eliminating q^* yields $Y = \frac{1}{g_0}\zeta^{1-\frac{\zeta}{\theta}}\left(\frac{\theta-\zeta}{k_0}\right)^{\frac{\zeta}{\theta}} K^{\frac{\zeta}{\theta}} L^{1-\frac{\zeta}{\theta}}$, which is a well-defined Cobb-Douglas production function given the proposition also shows $\theta > \zeta > 0$, $g_0 > 0$, $k_0 > 0$.

[34] See Wheeden and Zygmund (1977) Theorem 7.2 (p. 100), Theorem 7.11 (p. 107).

Proof of Proposition 3 We use the results from Lemma 4 proven below. Let $S^{-1}(q)$ denote the inverse of (2). **Part I.** By the way of contradiction, assume $\mu(\mathcal{Q}) < +\infty$. By Lemma 4, we must have $k(q) = CS(q)^{-\frac{1}{\alpha}}$ for some constant $C > 0$ because $S(\varepsilon) \to_{\varepsilon\to 0} S_0 < \infty$, or $k(q) = 0$ for all q. The latter case is not possible, and the former case implies $k(q) \to_{q\to 0} C_0 > 0$, by the fact that $\sup_{q\in\mathcal{Q}} S(q) = \mu(\mathcal{Q}) < +\infty$. But this contradicts Lemma 4 because that lemma implies that CD function must feature $k(q) \to_{q\to 0} 0$. Concluding, $\mu(\mathcal{Q}) = \infty$. **Part II.** Define a mapping $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f(x) := S^{-1}(Ax^{-1})$, where $A > 0$ is a scalar. Note that f obeys the following properties: i) f is strictly increasing by the fact that S is a survival functions, and hence it is strictly decreasing (note that we are in $\mathcal{T}^{ext}$ and density has full support); ii) $f(x) \to_{x\to\infty} \infty$ by the fact that $x^{-1} \to_{x\to\infty} 0$, and S is a strictly decreasing function with the property $S(x) \to_{x\to\infty} 0$; iii) $f(x) \to_{x\to 0} \infty$ by Part I given S aggregates to a Cobb-Douglas production function ($\mu(\mathcal{Q}) = \infty$ implies $S(q) \to_{q\to 0} \infty$). From Lemma 4, we know we can represent capital requirement by $k(q) = C_\varepsilon S(\varepsilon)^{\frac{1}{\alpha}} S(q)^{-\frac{1}{\alpha}} = C_\varepsilon \varepsilon^{-\frac{1}{\alpha}} q^{\frac{1}{\alpha}}$, for all $q \geq \varepsilon$. Without a loss, we can define scalars Z, α such that $C_\varepsilon \varepsilon^{-\frac{1}{\alpha}} = Z^{-1}\frac{1-\alpha}{\alpha}$, for any fixed $\varepsilon \in \mathcal{Q}$, where these scalars are independent of ε because by that lemma for any ε,ε' we must also have $C_\varepsilon S(\varepsilon)^{\frac{1}{\alpha}} S(q)^{-\frac{1}{\alpha}} = C_{\varepsilon'} S(\varepsilon')^{\frac{1}{\alpha}} S(q)^{-\frac{1}{\alpha}}$, for any $q \geq \max\{\varepsilon, \varepsilon'\}$, and $S(q) > 0$. To obtain the transformation in terms of the survival function as stated, note that $S(f(q)) = \frac{1}{q}$. We can also calculate the new density by evaluating $-\frac{d}{dq} S(f(q))$ (a.e.), which gives $-\frac{d}{dq}\left(S\left(S^{-1}\left(q^{-1}\right)\right)\right) = -\frac{d}{dq}\left(q^{-1}\right) = q^{-2}$. The transformation f thus yields $\left(\mathbb{R}_+, k(q) = Z^{-1}\frac{1-\alpha}{\alpha} q^{\frac{1}{\alpha}}, Aq^{-2}\right) \equiv T^{CD}$ (a.e., for functions). The fact that T^{CD} aggregates to the same CD function has been shown in Example 1 in text. Q.E.D.

Proof of Lemma 4 **Part I.** Necessity: If the relationship between Y, K and L is Cobb-Douglas with exponent α ($Y = AK^\alpha L^{1-\alpha}$, where $0 < \alpha < 1, A > 0$), the relationship between factor intensities is $\left(\frac{K}{Y}\right)^\alpha \left(\frac{L}{Y}\right)^{1-\alpha} \equiv \frac{1}{A}$, for some constant $A > 0$. Plugging in for factor intensities from (17), gives $\left(\int_0^{q^*} k(q)\, g(q)\, dv\right)^\alpha S(q^*)^{1-\alpha} \equiv \frac{1}{A}$. By Lebesgue's differentiation theorem,[35]

[35] See Wheeden and Zygmund (1977) Theorem 7.2 (p. 100), Theorem 7.11 (p. 107), and the comment under the proof of Theorem 7.16 (p. 109). The derivative of the left-hand side is $\lim_{r\to 0} \left|\frac{\int_{[q_0,q^*+r]} k(q)g(q)dv - \int_{[q_0,q^*]} k(q)g(q)dv}{r}\right| = \lim_{r\to 0}\left|\frac{\int_{[q^*,q^*+r]} k(q)g(q)dv}{r}\right| = k(q)\, g(q)$ (a.e.), where the first equality follows from Theorem 5.24 and the last equality follows from Theorem 7.2, as referenced.

the derivative with respect to q^* is

$$\alpha\left(\int_0^{q^*} k(q)\,g(q)\,dv\right)^{\alpha-1} k(q^*)\,g(q^*)\,S(q^*)^{1-\alpha}$$
$$-(1-\alpha)\left(\int_0^{q^*} k(q)\,g(q)\,dv\right)^{\alpha} S(q^*)^{-\alpha}\,g(q^*) \equiv 0,$$

a.e., which simplifies to

$$\int_0^{q^*} k(q)\,g(q)\,dv \equiv \frac{\alpha}{1-\alpha} k(q^*)\,S(q^*)\,. \tag{33}$$

The original equation needs to hold up to an arbitrary constant $A > 0$, and so we do not lose sufficiency by differentiating the original expression. We differentiate the above equation again with respect to q^*. The involved functions on the right-hand side, note, are monotone, implying they are differentiable a.e.[36] Differentiation gives $k(q)\,g(q) = \frac{\alpha}{1-\alpha}k'(q)\,S(q) - \frac{\alpha}{1-\alpha}k(q)\,g(q)$,where the asterisk over q is no longer needed. Simplifying, we obtain $\alpha\frac{k'(q)}{k(q)} = h(q) := \frac{g(q)}{S(q)}$. By Assumption 1 S is finite-valued on any interval $[q,\infty)$, where $q > 0$ (see the proof of Lemma 1). The ODE holds a.e. and it is only a necessary condition. We relegated the rest of the proof to the Online Appendix B. (Part II of the proof derives sufficiency conditions in the lemma and Part III solves the implied ODE as stated. We omit Part II from here because the stated solution of the ODE can be easily verified.) Q.E.D.

Proof of Proposition 4 Note that the fixed point in (21) implies

$$Pk(q) := Z^{-1}q^{\frac{1}{\alpha}}\left(\int_0^{q^*} rk(q)\,\tilde{g}^0(q)\,dq + w\tilde{S}^0(q^*)\right), \tag{34}$$

where $\tilde{S}(q^*) = \int_{q^*}^{\infty} \tilde{g}(q)\,dq$ is the survival function associated with the base density $\tilde{g}$. Recall from the firm cost minimization that the cutoff q^* is $q^* \equiv q^*\left(\frac{w}{r}\right) = k^{-1}\left(\frac{w}{r}\right)$, which yields

$$\tilde{S}(q^*) \equiv \tilde{S}\left(q^*\left(\frac{w}{r}\right)\right) = \tilde{S}\left(k^{-1}\left(\frac{w}{r}\right)\right). \tag{35}$$

[36] Theorem 7.21 in Wheeden and Zygmund (1977).

For the time being, assume that the fixed point $k(q)$ is such that its inverse is well defined. We will return to this. By Lemma 4 (see equation (18)), we know that the requirement that the production function in the capital q producing sector is Cobb-Douglas (any q) implies[37]

$$\int_0^{q^*} rk(q)\,\tilde{g}(q)\,dq \equiv \frac{\alpha}{1-\alpha} rk(q^*)\,\tilde{S}(q^*)\,. \tag{36}$$

Substituting (35) and (36) into (34), we obtain $Pk(q) = Z^{-1} q^{\frac{1}{\alpha}} \tilde{S}\left(k^{-1}\left(\frac{w}{r}\right)\right) \frac{w}{1-\alpha}$, where $\tilde{c}(w,r) := \tilde{S}\left(q^*\left(\frac{w}{r}\right)\right) \frac{w}{1-\alpha}$. We have now shown that the capital requirement function $k(q)$ is as stated up to the term $\tilde{S}\left(k^{-1}\left(\frac{w}{r}\right)\right) \frac{w}{1-\alpha}$, which is a constant from the firm's point of view. To see it is unit cost as stated, note that $L_q := \tilde{S}\left(q^*\left(\frac{w}{r}\right)\right)$ is the total labor input into the unit production of capital of type q by such a firm, and note that we can rewrite the previous equation as $\tilde{c}(w,r)(1-\alpha) = L_q w$. Since we have assumed that the production function in the capital producing sectors is Cobb-Douglas with common share parameter α, by definition of α the last expression implies that $\tilde{c}(w,r)$ must the total unit cost of that firm because $L_q w$ is the total labor cost. Concluding, $k(q) = Z^{-1} q^{\frac{1}{\alpha}} \frac{\tilde{c}(w,r)}{P}$. Following the approach in Example 1, it is straightforward to derive the production function. Q.E.D.

Proof of Proposition 5 The first part of the proof is in text. Observe that the breakthrough technology is applied to capital producing task for all q such that $\frac{b}{\beta}\left(\frac{\kappa_0}{b} q^{\frac{1}{\alpha}}\right)^{\frac{1}{1+\beta}} \left(\left(\frac{\kappa_0}{b}\right)^{\frac{1}{1+\beta}} + 1\right) \leq q^{\frac{1}{\alpha}}$. It is clear from these functional forms that there exists a finite cutoff $q_{\min}$ such that the technology is applied for all $q \geq q_{\min}$ ($\beta > 0$). We know $k(q) = q^{\frac{1}{\alpha}}$ for $q \leq q_{\min}$ and $k(q) = Cq^{\frac{1}{\alpha}(1+\beta)}$, otherwise, where the constant of proportionality C is such that there is no discontinuity at q_{min} between these two pieces, as implied by the continuity of the outer min operator in (23). We thus have: $\frac{K}{Y} \propto \left(\int_0^{q_{\min}} q^{\frac{1}{\alpha}-2} dq + C\int_{q_{\min}}^{q^*} q^{\frac{1}{\alpha}(1+\beta)-2} dq\right)$, for some constant $C > 0$ so that the above function is continuous at q_{min} as required. Define the isoquant error as the difference between isoquants relative to the counterfactual $k(q) = Cq^{\frac{1}{\alpha}(1+\beta)}$ but applied globally (to all q); that is, let $\frac{K'}{Y'} = C\int_0^{q^*} q^{\frac{1}{\alpha}(1+\beta)-2} dq$, for the same constant C.

[37] To adopt this lemma to the production of capital, we associate output Y with the number of units of capital that are produced in the capital sector q and K, L with the total factor usage in the sector that produces capital of type q.

After manipulations, it can be shown that obtain[38]

$$\varepsilon := \frac{|\frac{K}{Y} - \frac{K'}{Y'}|}{\frac{K'}{Y'}} = |\left(\frac{q_{\min}}{q^*}\right)^{1-\frac{1}{\alpha}} \frac{\left(C\left(\frac{q_{\min}}{q^*}\right)^{\frac{\beta}{\alpha}}(1-\alpha) - (q^*)^{-\frac{\beta}{\alpha}}(1-\alpha+\beta)(1+\beta)\right)}{(\alpha-1)}|.$$

Accordingly, $\varepsilon \to 0$ as $\frac{q_{\min}}{q^*} \to 0$, and if q^* is non-decreasing (economy automates). Consequently, the post-transformation isoquant converges uniformly to an CD isoquant and so does the labor share. The (aggregate) isoquant error is what is relevant for the aggregate labor share. Q.E.D.

[38]Explicit derivation of this formula is in the Mathematica notebook *Proposition_breakthrough.nb* posted online.