Understanding Growth Through Automation: The Neoclassical Perspective*

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Abstract

Technological progress enables capital to automate an ever-growing range of production tasks, yet the labor share of income has remained remarkably stable. To explain this puzzle, we develop a model in which production requires completing tasks using capital or labor. Over time, technological improvements make capital increasingly productive, and the mass of tasks assigned to labor shrinks toward zero. The labor share, however, converges to a positive constant, sustained by a thick tail of hard-to-automate labor tasks. In the limit, the task-based technology converges to a Cobb–Douglas production function. Embedded in a neoclassical growth model, the economy exhibits a pre-industrial regime with no capital accumulation, an endogenous industrial takeoff, and convergence to a balanced growth path. Occupation-level automation probabilities support the model's distributional predictions and imply a labor share in the range observed in US postwar data.

Keywords: Automation, labor share, Cobb-Douglas production function, capital-augmenting technological progress, balanced growth

JEL Classifications: D33, E25, O33.

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1 Introduction

Discovering new ways to perform work using capital instead of labor is a hallmark of technological progress. From steam engines replacing physical exertion during the Industrial Revolution to algorithms automating cognitive tasks today, the progressive automation of production has been a central driver of economic growth. Yet, as capital takes over an ever-increasing set of tasks, one might expect the share of national income going to labor to decline. Instead, as Keynes (1939) famously observed, it is "a bit of a miracle" that the labor share has remained so stable over time.[1] In this paper, we propose a task-based model of production that resolves the apparent tension between micro-level labor displacement and macro-level stability. In our framework, the set of automated tasks expands continuously over time. Nevertheless, the production technology converges to a Cobb–Douglas function, ensuring stable factor shares even in the presence of ongoing automation.

In the model, production requires completing a continuum of tasks, each of which can be performed using either capital or labor. Each task has a relative capital requirement, which captures the amount of capital needed, relative to labor, to complete it. That requirement evolves stochastically over time as innovations arrive. We assume that these innovations are proportional: halving the capital requirement of a complex task is no easier than halving that of a simple one. Combined with an average decline in capital requirements over time—consistent with the secular fall in the relative price of investment goods—this process generates a Pareto distribution of capital requirements across tasks (Gabaix, 2009): while most tasks can be performed efficiently by capital, a thick tail of highly complex tasks resists automation.

A cost-minimizing firm automates all tasks where capital is sufficiently efficient and assigns the remainder to labor. As technology improves, capital takes over an ever-growing range of tasks, and the labor share is governed by two opposing forces. On the extensive margin, the fraction of tasks performed by labor shrinks, pushing the labor share down. On the intensive margin, the tasks already automated become increasingly cheap to perform, pushing the labor share up. The Pareto distribution ensures that these two forces balance exactly. As a result, the labor share converges to a strictly positive constant, and the production technology converges to a Cobb–Douglas function. The long-run labor share is pinned down by the thickness of the Pareto tail, which governs how quickly labor tasks become rare, and the degree of complementarity across tasks, which determines how costly the remaining labor bottlenecks are.

We formalize the link between the capital requirement distribution and the labor share in a general result that extends beyond the Pareto case. The relevant object is the tail of that distribution. When the tail is thin, as in the exponential or normal cases, complex outliers are rare. Capital eventually automates essentially the entire economy and the labor share collapses to zero. Con-

[1] While more recent evidence suggests that the labor share has started to decline in many countries since the 1980s (Karabarbounis and Neiman, 2014), this decline follows a remarkably long period of stability.

versely, when the tail is very thick, the task space is dominated by tasks that are nearly impossible to automate, and the labor share converges to one. A stable, interior labor share obtains only when the tail elasticity falls in the intermediate range of scale-free power-law distributions. This is not a knife-edge condition: any distribution whose tail decays at a power-law rate within a well-defined interval delivers the same qualitative result. The Pareto, Fréchet, and other families belong to this class.

Because the labor share is governed by the tail of the complexity distribution, the model also provides a framework for understanding why it might change. The decline observed since the early 1980s can be interpreted as a thinning of the Pareto tail: if innovation accelerates or becomes more systematic, as plausibly occurred during the ICT revolution, the tail index rises and the labor share settles at a new, lower level. Artificial intelligence may represent a qualitatively different force. Previous waves of automation primarily reduced capital requirements for tasks near the technological frontier: routine manual and cognitive operations that were already relatively easy to mechanize. AI, by contrast, targets tasks that were previously deep in the thick tail: cognition, judgment, and contextual reasoning. In the language of the model, AI compresses the tail itself rather than merely advancing the frontier. If the tail loses its scale-free character, the labor share need not stabilize at all.

We embed the task-based technology in an otherwise standard neoclassical growth model to study its dynamic implications. Since the technology converges to Cobb–Douglas, the economy converges to a balanced growth path along which output, capital, and consumption all grow at a common rate determined by the pace of technological progress. More distinctive are the model's implications away from the long run. Early in the development process, capital is too inefficient to profitably automate even the simplest task. The economy is then trapped in a pre-industrial regime: a pure-labor state with no capital accumulation and no growth. The model therefore provides a simple rationale for the prolonged stagnation that preceded the modern growth era. As the technology frontier gradually improves, the marginal product of capital eventually exceeds the household's required rate of return, triggering the onset of industrialization. Once this threshold is crossed, reinforcing dynamics take over: automation raises labor productivity and wages, which in turn increases the incentive to automate further, propelling the economy into the modern growth era. We characterize the full transition dynamics between these two regimes.

We bring the model to data using the occupation-level automation probabilities constructed by Frey and Osborne (2017), who assign to each of 702 US occupations a probability of computerization over the next one to two decades based on expert assessments of task content. These probabilities provide a direct window into the cross-section of automation exposure, which is precisely the object that the model's Pareto tail governs. Under the interpretation that each occupation performs a representative task, the model generates a specific prediction for the shape of this cross-section: occupations should be distributed according to an exponential distribution of distances from the automation frontier. We show that this prediction fits the data reasonably well. For innovation rates consistent with the quality-adjusted decline in equipment prices, the implied labor share is approximately 0.59, in the range observed in US postwar data. This concordance between micro-level automation exposure and macro-level factor shares is reassuring and provide support for our task-based model of production.

Literature review Our paper builds on the task-based approach to automation and growth as in Zeira (1998) and Acemoglu and Restrepo (2018). In the framework of Acemoglu and Restrepo (2018), automation displaces labor from existing tasks, reducing the labor share, but new labor-intensive tasks are continuously created to offset this displacement. Balanced growth obtains when the rate of task creation matches the rate of automation, so that the fraction of automated tasks remains constant over time. In our framework, instead, the set of tasks is fixed: no tasks are created or destroyed, and yet the labor share converges to a positive constant. The stabilizing force arises from the power-law distribution of the task complexity distribution, which ensures that the cost savings from automated tasks exactly offset the shrinking mass of labor tasks. As we show, the stable labor share in our model holds for any innovation process that generates a distribution of task complexity with Pareto-like tail.

Also related is Jones and Liu (2024), who develop a task-based model in which capital advances on two margins: automation (taking over new tasks) and productivity improvement (becoming more efficient at existing tasks). These two forces engage in a tug-of-war that can produce balanced growth despite capital-embodied technological progress. A key difference is that in their baseline model these two margins are governed by independent technology processes, and balanced growth requires these processes to advance at the same rate. In our model, both margins arise from a single innovation process: the same stochastic decline in capital requirements that automates new tasks also makes already-automated tasks cheaper. The balance between the two forces is then a property of the Pareto distribution, not a restriction on exogenous rates. A further distinction is that their aggregate production function maintains a non-unitary elasticity of substitution throughout, satisfying the Uzawa (1961) theorem by holding the capital-augmenting term constant. In our framework, the production function converges to Cobb–Douglas—a different resolution of the same tension.

A separate strand of the literature explains labor-share stability through the Baumol (1967) cost-disease mechanism operating across sectors or goods with different rates of productivity growth (Ngai and Pissarides, 2007; Aghion et al., 2019). In these models, goods produced by fast-improving sectors become relatively cheap, and spending shifts toward slow-improving sectors, preventing any single factor from dominating the cost of production. Our model features a version of this mechanism, but it operates at the task level rather than the sectoral level. As capital becomes more efficient at automated tasks, their cost falls and their weight in total expenditure declines, following the Baumol logic. The key difference is that in our framework, this rebalancing force does not require assumptions on the elasticity of substitution across final goods or on the relationship between sectoral productivity growth rates. Instead, it emerges endogenously from the power-law distribution of task-level capital requirements.

Our model also relates to the literature on the transition from stagnation to growth. Hansen and Prescott (2002) model a regime switch in which a capital-intensive "Solow" technology coexists with a labor-intensive "Malthusian" technology; growth begins when exogenous TFP improvements make the Solow sector profitable. Our mechanism shares the broad logic but differs in that the transition arises endogenously from a single task-based production structure rather than from a discrete switch between two exogenously specified production functions. The unified growth theory of Galor (2005) emphasizes Malthusian forces and demographic transitions as drivers of the stagnation-to-growth transition. Our mechanism is complementary: stagnation arises not from population pressure but from the bounded return to capital when even the simplest task has a finite capital requirement.

Finally, our paper contributes to the literature on the microfoundations of the Cobb–Douglas production function. Houthakker (1955) showed that aggregating over a population of capacity-constrained Leontief production units with Pareto-distributed input coefficients yields a Cobb–Douglas production function. Jones (2005) shifted the mechanism from aggregation across firms to technique choice within a single production unit: if a firm selects optimally from a menu of techniques whose parameters are drawn from a Pareto distribution, the resulting global production function is Cobb–Douglas with constant returns to scale, and technical change is labor-augmenting in the long run. In both cases, the Pareto distribution is assumed as a primitive. Our paper provides a third, complementary mechanism: dynamic task allocation. n our model, the Pareto distribution is not assumed but derived from the scale-free stochastic evolution of task-level capital productivity. Moreover, our result is inherently dynamic: Cobb–Douglas production obtains only in the limit as the economy matures, while the technology deviates from Cobb–Douglas during the transition.

Outline The remainder of the paper is organized as follows. Section 2 introduces the task-based production model and derives the stationary distribution of task complexity. Section 3 derives the aggregate production function, establishes convergence to Cobb–Douglas, and characterizes the role of the complexity tail. Section 4 embeds the technology in a neoclassical growth model and analyzes the balanced growth path, the pre-industrial trap, and the transition dynamics. Section 5 uses occupation-level automation probabilities to assess the model's distributional predictions and estimate the Pareto tail index. Section 6 concludes.

2 A task-based model of production

We embed a task-based model of production into an otherwise standard neoclassical growth framework. Production requires completing a continuum of tasks, each of which can be performed using capital or labor. The key object is the relative capital requirement of each task, which captures how much capital is needed to replace one unit of labor. This requirement evolves stochastically over time as innovations arrive, giving rise to a Pareto distribution of capital requirements across tasks.

2.1 Neoclassical growth framework

Time is continuous with $t \in [0, \infty)$. A representative household supplies $L > 0$ units of labor inelastically and values consumption $\{C_t\}_{t \geq 0}$ according to the utility function

$$\int_0^\infty e^{-\rho t} \log(C_t)\, dt,$$

where $\rho > 0$ is the discount rate.

Aggregate output Y_t at time t is produced using labor L and capital K_t through a technology that we describe below. Output can be used for either consumption C_t or investment $Y_t - C_t$ in the capital stock. Capital depreciates at a fixed rate $\delta > 0$, and the capital stock evolves according to

$$\dot{K}_t := dK_t/dt = Y_t - C_t - \delta K_t. \tag{1}$$

To keep the exposition transparent, we assume log utility, abstract from population growth, and omit labor-augmenting technical progress, though extending to CRRA preferences and incorporating these features is straightforward. When it is not confusing, we drop the time subscript t to lighten the notation.

2.2 Task technology

A firm must complete a unit mass of tasks indexed by $i \in [0, 1]$ to produce output. We think of a task as the most elementary operation required for production. For instance a task could involve welding a joint, computing a payroll, or sorting a package. Producing one unit of output requires

$$\left(\int_0^1 (a\,(i))^{\frac{\gamma-1}{\gamma}}\; di \right)^{\frac{\gamma}{\gamma-1}} \geq A, \tag{2}$$

where $a\,(i) \geq 0$ is the intensity at which task i is performed and $A > 0$ is the required aggregate intensity.

Assumption 1 (Task-level Elasticity)**.** *The elasticity of substitution across tasks satisfies* $0 \leq \gamma < 1$.

The parameter γ governs the degree of substitutability across tasks. When $\gamma > 0$, a firm can partially compensate for low intensity in one task by increasing the intensity of others. For instance, a restaurant that finds one ingredient scarce can adjust the composition of its dish. The restriction $\gamma < 1$ means that tasks are complements: increases in the cost of certain tasks cannot be fully offset by reallocating intensity to other tasks. As in the classical Baumol cost-disease mechanism, tasks with slower productivity growth then become disproportionately expensive. This complementarity will play an important role in sustaining the labor share. When $\gamma \to 0$, the technology reduces to the Leontief case in which every task must be completed at a uniform intensity $a(i) = A$.

Each task can be completed using either capital or labor, but not a combination of both. Completing task i at intensity $a(i)$ takes $a(i)$ units of labor or $q_t(i)\,a(i)$ units of capital.[2] The quantity $q_t(i) > 0$ is the *relative capital requirement* of task i: a low value means that capital can perform the task efficiently, while a high value means the task is difficult to automate. As we will show in Section 3, a cost-minimizing firm uses capital for tasks with low q and labor for the rest, with the dividing line determined by the ratio of the wage to the rental rate.

2.3 Stochastic evolution of capital requirements

We model technological progress as a stochastic process that gradually reduces the capital requirements of each task. The central assumption is that innovations are proportional: they build on prior advances, so that a task that has already seen substantial improvements in capital efficiency is just as likely to see further gains as one that has not. This scale-free property—capturing the idea that halving the capital requirement of a complex task is no easier than halving that of a simple one—leads us to model q_t as a geometric Brownian motion.

Assumption 2 (Task-level Innovation Process)**.** *For each task i, the relative capital requirement $q_t := q_t(i)$ evolves according to*

$$d \log q_t = -g dt + \sigma dW_t, \tag{3}$$

where $g > 0$ is the average rate of improvement and W_t is a standard Wiener process. Moreover, we assume that

$$q_t \geq q_0(t) := \exp(-\phi g t), \phi \in [0, 1),$$

so that the lower bound of the support declines over time. Each task becomes obsolete at a Poisson rate $\chi \geq 0$, independently across tasks. Upon obsolescence, the task is instantaneously reborn at the technological frontier, i.e. $q_{new}(t) = q_0(t)$.

The drift $g > 0$ captures the average pace at which engineers and entrepreneurs learn to perform a given task with capital rather than labor. The volatility σ captures the unpredictability of

[2] We model the evolution of the *relative* requirement q_t directly, but one could equivalently define independent stochastic processes for labor and capital requirements, q_t^l and q_t^k. Since the firm's choice depends only on the relative input requirements, the resulting dynamics for $q_t := q_t^k / q_t^l$ would be isomorphic to our baseline setup.

this process: some tasks see rapid automation (ATMs replacing bank tellers), while others stall for decades despite sustained investment (autonomous driving). The moving lower bound $q_0(t)$ represents the technological frontier: the minimum amount of capital needed to perform even the simplest task at any point in time. As technology improves, this frontier declines at rate ϕg. The hazard rate $\chi > 0$ introduces turnover: even tasks with mature technologies occasionally experience radical disruptions that reset their capital requirement to the frontier.

2.4 Pareto distribution of capital requirements

Our first formal result shows that the cross-sectional distribution of capital requirements converges to a Pareto distribution. The proof relies on standard stochastic-calculus arguments (Luttmer, 2007; Gabaix, 2009).

Lemma 1 (Stationary Distribution)**.** *Let $x_t := q_t e^{\phi g t}$ denote the capital requirement normalized by the technological frontier. The variable x_t admits a stationary Pareto distribution with probability density*

$$p(x) = \zeta x^{-\zeta-1}, \quad x \geq 1,$$

where the tail index ζ is given by

$$\zeta := \frac{g(1-\phi) + \sqrt{g^2(1-\phi)^2 + 2\sigma^2\chi}}{\sigma^2} \xrightarrow[\chi \to 0]{} 2g\frac{1-\phi}{\sigma^2}. \tag{4}$$

Equivalently, the unnormalized capital requirement q_t evolves according to a process with a moving support $q_t \geq q_0(t)$ and probability density

$$f_t^*(q) := \zeta q_0(t)^\zeta q^{-\zeta-1}, \quad q \geq q_0(t) := e^{-\phi g t}. \tag{5}$$

The distribution of capital requirements becomes stationary only when normalized by the technological frontier $q_0(t)$. The unnormalized distribution of q_t therefore shifts alongside the frontier over time. The bulk of the probability mass concentrates near the frontier, where the most efficient tasks reside, but the distribution features a fat Pareto tail extending to tasks with much higher capital requirements—tasks requiring cognition, sensory feedback, or small production runs. Because of those tasks, labor remains an essential factor of production in the long run.[3]

The thickness of the Pareto tail is governed by ζ. In the limit as task turnover vanishes ($\chi \to 0$), equation (4) shows that ζ depends on three features of the environment. First, a lower drift g produces a thicker tail (lower ζ), as tasks gravitate more slowly toward the efficiency frontier. Second, higher volatility σ^2 also thickens the tail, as larger shocks disperse tasks into the high-

[3]Pareto distributions are a robust empirical regularity in the natural and social sciences, governing phenomena such as the size distribution of cities, firms, and incomes (Gabaix, 2009).

requirement region. Third, a faster-moving frontier (higher ϕ) thickens the tail as well: tasks fall behind the frontier more easily when it advances rapidly.

The Pareto distribution is a classic outcome of proportional random growth processes (Gabaix, 2009). To obtain a stationary distribution, some friction is necessary to prevent the probability mass from escaping to zero. In our model, the moving lower bound $q_0(t)$ serves this purpose. Economically, this bound captures the idea that even the most capital-efficient task faces a minimum resource requirement set by the current state of technology. No task can be completed with zero inputs. From now on, we work with the limiting distribution as $\chi \to 0$.

3 Production function

In this section, we derive the production function implied by our task-level technology. We proceed in three steps: first, we characterize the optimal allocation of tasks to capital and labor; second, we aggregate these choices to derive the demand for factors; and finally, we show that as the technology frontier advances, the economy converges to a Cobb-Douglas aggregate production function with stable factor shares.

3.1 Optimal task allocation

Consider a firm endowed with the task technology (2) and that seeks to minimize its cost of production. The firm takes as given the prices of labor (wage w) and capital (rental rate r), and must decide to which factor to allocate each task.[4] For a given intensity $a(i)$, using labor to complete a task i costs $a(i)w$, while using capital costs $a(i)rq(i)$. It follows that the firm adopts a cutoff rule and uses capital for tasks where the relative capital requirement is low $rq(i) < w$, and labor otherwise. This defines a automation threshold

$$q^* := \frac{w}{r},$$

such that tasks with $q < q^*$ are automated with capital, while tasks with $q \geq q^*$ are performed by labor. If prices are such that $q^* \leq q_0$, even the task that is the easiest to automate is still produced with labor and so the firm uses no capital.

Throughout the analysis, we maintain the following parameter restriction:

$$\sigma^2 (1-\gamma) > 2(1-\phi)\, g, \tag{6}$$

or equivalently $\zeta < 1-\gamma$. This condition requires that the Pareto tail of the capital requirement distribution be sufficiently thick relative to the degree of task complementarity. It ensures that

[4] Alternatively, one can consider a firm that has access to L units of labor and K units of capital. In that case, w and r are the shadow prices of these two factors. The problem of a social planner can be handled in a similar way.

enough tasks remain costly to automate so that labor retains a positive share of production costs in the long run.

Given this allocation, the firm chooses the intensity profile $a(i)$ to minimize total cost subject to the production constraint (2). The solution of that problem leads to the following lemma.

Lemma 2. *Cost minimization implies a cutoff rule $q^* = w/r$. The optimal task intensity satisfies:*

$$a(q) = \begin{cases} a(q^*) \left(\frac{q}{q^*}\right)^{-\gamma} & \text{for } q < q^* \quad \text{(capital tasks)} \\ a(q^*) & \text{for } q \geq q^* \quad \text{(labor tasks)} \end{cases} \tag{7}$$

where $a(q^*)$ is the intensity at the cutoff, which is given by

$$a(q^*) = A \left(\frac{1-\gamma}{1-\gamma-\zeta} \left(\frac{q_0}{q^*}\right)^{\zeta} - \frac{\zeta}{1-\gamma-\zeta} \left(\frac{q_0}{q^*}\right)^{1-\gamma} \right)^{\frac{\gamma}{1-\gamma}}.$$

The intensity profile $a(q)$ reflects the firm's ability to substitute across tasks. For tasks assigned to labor ($q \geq q^*$), the marginal cost of intensity is fixed at the wage w, so the firm operates all such tasks at a constant baseline intensity $a(q^*)$. In contrast, for tasks assigned to capital ($q < q^*$), the marginal cost rq is strictly increasing in q. Since the firm is indifferent between inputs at the cutoff q^*, but faces strictly lower costs for infra-marginal capital tasks ($q < q^*$), it increases the intensity of these cheaper tasks whenever cross-task substitution is possible ($\gamma > 0$). Capital tasks are therefore always performed at a higher intensity than labor tasks. Figure 1 illustrates the task intensity schedule (red) together with the underlying distribution of capital requirements (blue).

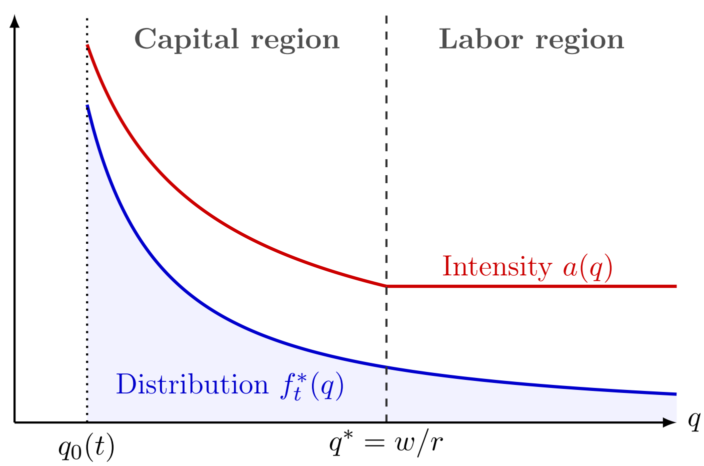

Figure 1: Task distribution and optimal intensity. The blue curve represents the stationary distribution of task capital requirements (Pareto tail). The red curve shows the optimal intensity profile. To the left of q^*, tasks are automated with capital and performed at high intensity ($a(q) > a(q^*)$). To the right, labor tasks are performed at a constant baseline intensity $a(q^*)$.

Finally, Lemma 2 provides an explicit expression for the intensity $a(q^*)$ of the labor tasks. That intensity is determined by the task intensity constraint (2). For a firm that uses labor exclusively ($q^* \leq q_0$), all tasks operate at the uniform intensity $a(q^*) = A$. As labor becomes relatively expensive ($q^* > q_0$), the firm shifts some tasks to the high-intensity capital factor. The contribution of these automated tasks relieves the aggregate production burden, allowing the firm to lower the intensity of the remaining labor tasks. Thus, as capital becomes a more important factor of production (higher q^*), the baseline labor intensity $a(q^*)$ declines.

Leontief limit. In the limit as $\gamma \to 0$, Lemma 2 takes on a particularly simple form and $a(q) = A$ for all tasks q. Without substitutability, the firm cannot shift production toward cheaper tasks.

3.2 Task aggregation

Using the cutoff q^* and the task intensity profile $a(q)$, we can compute the factor demand of the firm.

Lemma 3. *Per unit of output, the firm's optimal demand for labor L is given by*

$$\frac{L}{Y} = \underbrace{a(q^*)}_{\substack{\text{labor requirement of}\\ \text{labor tasks}}} \times \underbrace{\left(\frac{q_0}{q^*}\right)^{\zeta}}_{\substack{\text{mass of}\\ \text{labor tasks}}}, \tag{8}$$

and its optimal demand for capital K is given by

$$\frac{K}{Y} = \underbrace{q^* a(q^*)}_{\substack{\text{capital usage}\\ \text{at cutoff } q^*}} \times \underbrace{\zeta \left(\frac{q_0}{q^*}\right)^{\zeta}}_{\substack{\text{task density}\\ \text{at cutoff } q^*}} \times \underbrace{\frac{1}{1-\gamma-\zeta}}_{\substack{\text{scaling from marginal}\\ \text{to total}}} \times \underbrace{D\left(\frac{q_0}{q^*}\right)}_{\substack{\text{truncation}\\ \text{adjustment}}}, \tag{9}$$

where

$$D\left(\frac{q_0}{q^*}\right) := 1 - \left(\frac{q_0}{q^*}\right)^{1-\gamma-\zeta},$$

is the effective automation depth.

The expression for labor demand is intuitive. Since all labor tasks are performed at a constant baseline intensity $a(q^*)$ (Lemma 2), total labor demand is simply this intensity multiplied by the mass of tasks assigned to labor. Capital demand (9) is less straightforward because automated tasks are heterogeneous: a task with a low capital requirement q uses little capital, while a task near the cutoff q^* uses much more. Total capital demand therefore requires integrating over the entire range of automated tasks $[q_0, q^*]$. The Pareto distribution simplifies this integral considerably. Because of its self-similarity, total capital demand is always proportional to the capital used at the marginal task q^*, with the factor of proportionality $(1-\gamma-\zeta)^{-1}$ depending only on the tail index and the degree of task complementarity. The final term, $D\left(q_0/q^*\right)$, corrects for the fact that the distribution is bounded below by the technological frontier $q_0 \to 0$. We call this term the *effective automation depth*: it measures how far into the tail the firm has automated. When $q_0 \to 0$, automation extends deep into the tail, $D \to 1$, and the correction vanishes.

3.3 Labor share

We can use the expressions from Lemma 3 to compute standard firm-level quantities. For instance, the share of labor in total production costs is given by

$$s_L := \frac{wL}{wL + rK} = \left[1 + \frac{\zeta}{1-\gamma-\zeta} D\left(\frac{q_0}{q^*}\right)\right]^{-1} \xrightarrow[q_0 \to 0]{} \frac{1-\gamma-\zeta}{1-\gamma}. \tag{10}$$

Early in the automation process, when capital is relatively inefficient (q_0 large), labor performs all tasks ($q^* < q_0$) and the labor share is one. As technology evolves (q_0 decreases), capital takes over a growing range of tasks, causing the labor share to progressively decline. In the long run, however, this process reaches a natural limit and the labor share stabilizes at a strictly positive level rather than vanishing to zero.

Why is the labor share not converging to zero? As the frontier $q_0 \to 0$ (for a fixed q^*), two opposing forces are at work.[5] On the extensive margin, the mass of tasks performed by labor, $(q_0/q^*)^\zeta$, shrinks toward zero, pushing the labor share down. On the intensive margin, the cost of automated tasks also shrinks: as $q_0 \to 0$, most capital tasks can be completed with almost no capital at all, so average capital spending per task drops. The Pareto distribution ensures that these two forces balance exactly, and the labor share converges to a positive constant. Two parameters govern this balance: the tail index ζ, which controls how quickly labor tasks become rare, and the complementarity parameter γ, which prevents the firm from substituting away from the expensive bottleneck tasks that remain. It is therefore natural that ζ and γ determine the long-run labor share, as (10) shows.

3.4 The importance of the complexity tail

The thickness of the capital requirement distribution is the key determinant of the long-run labor share. To show this, we study in this section the behavior of s_L under different distributions. To do so, let $x = q/q_0 \in [1, \infty)$ denote the normalized task complexity, and suppose that x is drawn from a general continuous distribution with survival function $\bar{P}(x)$ and density $p(x) = -\bar{P}'(x)$. Define the *asymptotic tail elasticity* of the survival function as

$$\lambda := \lim_{x \to \infty} \frac{xp(x)}{\bar{P}(x)}$$

[5] In the full dynamic model, we will see that $q^* = w/r$ grows over time, so that $q_0 \to 0$ implies $q_0/q^* \to 0$.

This elasticity λ measures how rapidly the mass of hard-to-automate tasks vanishes. For our baseline Pareto distribution, $\lambda = \zeta$. A distribution with a thinner tail (e.g., Exponential, Normal) has an elasticity that diverges ($\lambda = \infty$), while a distribution with a thicker tail (e.g., a slowly varying function like $1/\ln x$) has an elasticity that vanishes ($\lambda = 0$). The following result characterizes the link between λ and the labor share.

Proposition 1. *Consider the task-based production model where the normalized task complexity $x = q/q_0 \in [1, \infty)$ is drawn from a continuous distribution with survival function $\bar{P}(x)$. Suppose $\bar{P}(x)$ is regularly varying at infinity with index $-\lambda \leq 0$ (i.e., $\bar{P}(x) = x^{-\lambda} L(x)$ for a slowly varying function L), or is rapidly varying ($\lambda = \infty$). Suppose the aggregate factor ratio $K/L > 0$ is fixed, implying the optimal automation cutoff q^* is bounded strictly away from zero. As the technological frontier advances ($q_0 \to 0$), the aggregate labor share s_L satisfies:*

$$\lim_{q_0 \to 0} s_L = \begin{cases} 0 & \text{if } \lambda \geq 1 - \gamma, \\ \frac{1-\gamma-\lambda}{1-\gamma} & \text{if } 0 < \lambda < 1 - \gamma, \\ 1 & \text{if } \lambda = 0. \end{cases} \tag{11}$$

Proposition 1 extends the labor share result beyond the Pareto case. Recall from Section 3.3 that the long-run labor share is determined by a balance between two forces: the shrinking mass of labor tasks (which pushes the labor share down) and the falling cost of automated tasks (which pushes it up). The tail elasticity λ determines which force dominates.

When $\lambda \geq 1-\gamma$, the tail decays fast enough that hard-to-automate tasks become negligible. This class includes the Exponential, Normal, and Lognormal distributions, as well as Pareto distributions with a sufficiently high tail index. Capital eventually automates the economy and the labor share collapses to zero. Conversely, when $\lambda = 0$, the tail decays so slowly that the task space is dominated by tasks that are nearly impossible to automate. The labor share converges to one.

A stable, interior labor share requires $0 < \lambda < 1 - \gamma$: the tail must decay at a power-law rate, and slowly enough that the two forces exactly offset. This class includes the Pareto and Fréchet distributions with tail indices below $1 - \gamma$. As such, the stable labor share result is not knife-edge, and many distribution with appropriate tail behavior would deliver the result.

3.5 Discussion

Proposition 1 shows that the stability of the labor share depends on the tail of the capital requirement distribution and, therefore, on the structural properties of the innovation process. If those properties change, so does the labor share. This observation suggests interpretations of two features of the recent economic landscape.

The recent decline in the labor share. The labor share has declined in many countries since the early 1980s (Karabarbounis and Neiman, 2014). Within our framework, this decline could reflect a thinning of the Pareto tail, captured by an increase in ζ that weakens the Baumol bottleneck sustaining the labor share. From equation (4), the tail thins when the pace of innovation g rises or when its volatility σ^2 falls. Both channels have natural interpretations in terms of the ICT revolution, which plausibly accelerated the pace of automation while making innovation more systematic. In terms of Proposition 1, such changes push the tail elasticity λ upward, reducing the long-run labor share. As long as λ remains below $1-\gamma$, the labor share stabilizes at a lower but still positive level.

Artificial intelligence. The tasks populating the thick tail of the capital requirement distribution are arguably those involving cognition, judgment, and contextual reasoning. Arguably, those are tasks where prior technologies made little headway. In the language of the model, previous waves of automation primarily reduced q for tasks near the technological frontier without much affecting the tail. Artificial intelligence may be different: by targeting cognitive tasks directly, it could reduce capital requirements for tasks that were previously deep in the tail, compressing the tail itself rather than merely advancing the frontier $q_0(t)$. If so, AI would increase the tail elasticity λ, potentially pushing it above the $1-\gamma$ threshold at which point the labor share begins to decline toward zero. Whether this threshold is approached depends on how broadly AI can overcome cognitive barriers across the spectrum of tasks.

3.6 Illustrative example: two types of innovation

The labor share is shaped by two opposing forces: the intensive margin (automated tasks becoming cheaper) and the extensive margin (new tasks being automated). To build intuition for when each dominates, we work through a discrete example in the Leontief case ($\gamma = 0$).

Consider an economy with four tasks, $i \in \{1, 2, 3, 4\}$, each of which must be completed once per unit of output. Each task can be performed by one unit of labor (at cost w) or by $q(i)$ units of capital (at cost $rq(i)$). The capital requirements are

$$q(1) = 1, \quad q(2) = 2, \quad q(3) = 3, \quad q(4) = 4.$$

A cost-minimizing firm automates task i whenever $rq(i) \leq w$, i.e., whenever $q(i) \leq w/r$. On the balanced growth path, the rental rate r is pinned by the Euler equation and depreciation: $r = \rho + \delta + \nu$. We normalize $r = 1$ throughout. With constant returns to scale and free entry, the cost per unit of output equals its price. Since the price-to-rental ratio $P/r = 1/(\rho + \delta + \nu)$ is constant across steady states, the cost per unit of output (measured in capital service units) is the same in every steady state. In our example, this constant equals $\eta = 7$.[6]

Baseline. At $w/r = 2$, the firm automates tasks 1 and 2 (where $q \leq 2$) and assigns tasks 3 and 4 to labor. Per unit of output, the firm uses $K = q(1) + q(2) = 3$ units of capital and $L = 2$ units of labor. The zero-profit condition pins the wage:

$$\underbrace{K \cdot r}_{3} + \underbrace{L \cdot w}_{2\times 2=4} = \eta = 7.$$

The labor share is $s_L = Lw/\eta = 4/7 \approx 57\%$. Panel (a) of Figure 2 illustrates this allocation.

Example 1: intensive margin (an automated task gets cheaper). Suppose a technological improvement reduces the capital requirement of task 2 from $q(2) = 2$ to $q(2) = 1$. Since task 2 was already automated, the set of automated tasks does not change. But the capital needed per unit of output falls from $K = 3$ to $K' = 2$.

On impact, the cost of production drops to $2 + 4 = 6 < 7$, generating a profit of 1. Competition eliminates this profit: with r fixed and the cost per unit pinned at $\eta = 7$, the wage must rise to absorb the saving. The new zero-profit condition gives

$$2 \cdot 1 + 2 \cdot w = 7 \quad \Longrightarrow \quad w^* = \frac{5}{2}.$$

The new wage-rental ratio $w^*/r = 5/2$ is higher than before, but not high enough to trigger automation of task 3 (since $q(3) = 3 > 5/2$), so the task allocation is unchanged. The labor share rises to $s_L = 5/7 \approx 71\%$.

The economics are transparent: the rental rate is fixed on the balanced growth path, so capital owners cannot capture the cost saving. Competition forces the entire gain into higher wages. The labor share rises because the same number of workers now share the pie with less capital. Panel (b) of Figure 2 shows the new allocation.

Example 2: extensive margin (a labor task becomes automatable). Now return to the baseline and consider a different innovation: the capital requirement of task 3 falls from $q(3) = 3$ to $q(3) = 2$. At the original $w/r = 2$, task 3 is now weakly cheaper to automate than to perform with labor, and the firm automates it.

The task allocation changes: three tasks are now automated and only task 4 remains with labor. Per unit of output, $K' = 1 + 2 + 2 = 5$ and $L' = 1$. The zero-profit condition gives

$$5 \cdot 1 + 1 \cdot w = 7 \quad \Longrightarrow \quad w^* = 2.$$

[6] This normalization implies $\rho + \delta + \nu = 1/7$. The specific value does not matter; what matters is that η is the same across the steady states we compare.

The wage is unchanged. This is because task 3, at its new capital requirement $q(3) = 2$, costs exactly as much to automate ($rq = 2$) as it did to perform with labor ($w = 2$). There is no cost saving per unit of output—the innovation simply substitutes capital for labor on one task without changing the total cost.

But the labor share falls sharply, from 4/7 to $s_L = 2/7 \approx 29\%$. Why? Each unit of output now requires only one worker instead of two. With a fixed labor force, the economy produces twice as many units of output, but the additional output is produced with capital, not labor. Workers are not worse off in absolute terms—their wage is the same—but their share of the larger pie is smaller. Panel (c) of Figure 2 shows the new allocation.

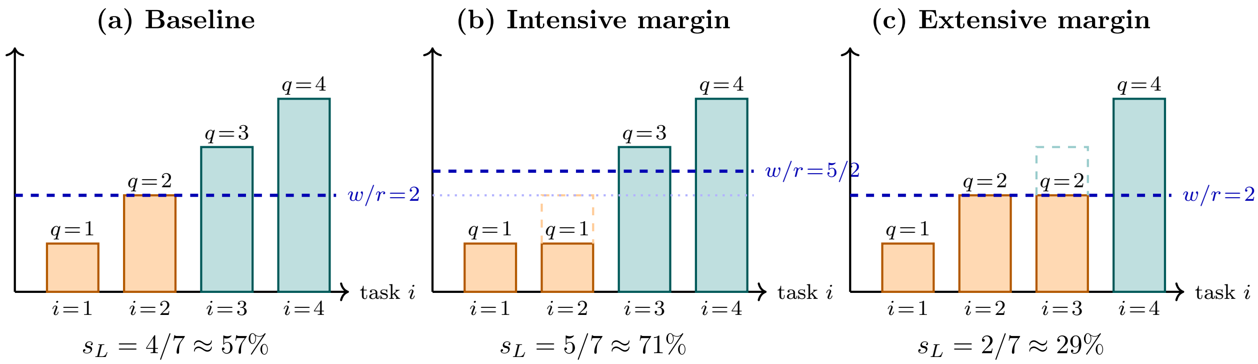

Figure 2: Two types of innovation in a four-task economy. Orange bars represent automated tasks; teal bars represent labor tasks. Dashed outlines show the pre-innovation values. The horizontal dashed line marks the automation cutoff w/r. Panel (a): baseline. Panel (b): an already-automated task becomes cheaper (intensive margin)—the cutoff rises, wages increase, and the labor share rises. Panel (c): a labor task becomes automatable (extensive margin)—the task switches from labor to capital, wages are unchanged, and the labor share falls.

Discussion. The two examples isolate the forces that govern the labor share in the continuous model. Innovation that makes already-automated tasks cheaper (the intensive margin) reduces capital costs per unit of output without changing the set of automated tasks. Since the rental rate is fixed on the balanced growth path, the entire cost saving is absorbed by higher wages, and the labor share rises. Innovation that brings a new task into the automated range (the extensive margin) replaces a labor task with a capital task, lowering the labor share. In the continuous model with a Pareto distribution of capital requirements, both forces operate simultaneously as the technological frontier $q_0(t)$ declines: existing automated tasks become cheaper, and new tasks cross the automation cutoff. As we showed in Section 3.3, the Pareto tail ensures that these two forces exactly offset in the long run, stabilizing the labor share. A thicker tail (lower ζ) means more hard-to-automate tasks, strengthening the intensive margin relative to the extensive margin and producing a higher labor share.

3.7 Elasticity of substitution between capital and labor

The task-based technology also has implications for the elasticity of substitution between capital and labor

$$\sigma_{KL} := \frac{d\ln(K/L)}{d\ln(w/r)}.$$

Lemma 4. *Suppose that $q^* > q_0$. The elasticity of substitution between capital and labor for the task-based technology is given by*

$$\sigma_{KL} = 1 + \frac{\beta \left(\frac{q_0}{q^*}\right)^{\beta}}{1 - \left(\frac{q_0}{q^*}\right)^{\beta}} \geq 1,$$

where $\beta = 1 - \gamma - \zeta > 0$. Consequently, the elasticity of substitution satisfies $\sigma \geq 1$ and converges to the Cobb-Douglas unitary value $\sigma \to 1$ as $q_0 \to 0$.

Early in the automation process, when labor performs all tasks ($q^* < q_0$), the elasticity of substitution between capital and labor is effectively zero. At this stage, the rental rate is too high relative to the simple frontier tasks, so no substitution occurs regardless of marginal price variations. However, once the technological frontier q_0 improves sufficiently that the first task becomes profitable ($q^* \approx q_0$), the elasticity spikes, and the input factors are gross substitutes ($\sigma_{KL} > 1$). This occurs because the distribution of automatable tasks is then highly concentrated near the frontier q^*. As a result, even a small increase in the wage relative to the rental rate leads to a disproportionately large mass of tasks switching from labor to capital. As the economy matures and automation deepens ($q^* \gg q_0$), this sensitivity declines. The threshold q^* moves into the tail of the Pareto distribution, where tasks are sparser and more complex. Marginal changes in factor prices induce smaller reallocations of tasks, causing the aggregate elasticity to fall. In the long-run limit ($q_0 \to 0$), the elasticity converges smoothly to the unitary value characteristic of the Cobb-Douglas production function.

3.8 Cobb-Douglas limit

Equation (10) shows that the labor share of the task-based technology converges to a constant that is independent of factor prices. As is well-known, a constant returns-to-scale Cobb-Douglas production function with labor elasticity

$$\alpha := \frac{1 - \gamma - \zeta}{1 - \gamma}$$

would also share the same property. In addition, the capital to labor ratio of the task-based technology also converges to its Cobb-Douglas counterpart. Indeed, one can show that

$$\frac{K}{L}=\frac{\zeta}{1-\gamma-\zeta}\frac{w}{r}\left[1-\left(\frac{q_0}{q^*}\right)^{1-\gamma-\zeta}\right]\xrightarrow[q_0\to 0]{}\frac{1-\alpha}{\alpha}\frac{w}{r}. \tag{12}$$

Since these properties hold for any $w>0$ and $r>0$, we can show that the task-based production function indeed converges to Cobb-Douglas. To establish this result, we first define this production function formally.

Definition 1 (Task-based Production Function)**.** Given capital and labor endowments K and L, as well as a technological frontier q_0, the *task-based production function* $F(K,L;q_0)$ is the maximum output Y achievable using the task-based technology.

The following result establishes the equivalence of the task-based and the Cobb-Douglas production functions in the long run.

Proposition 2. *Let $b>0$ and define $\mathcal{D}_b=\{(K,L):K/L>b\}$. As $q_0\to 0$, the task-based production function $F(K,L;q_0)$ converges on $\mathcal{D}_b$ to the Cobb-Douglas limit uniformly in relative error. That is*

$$\lim_{q_0\to 0}\sup_{(K,L)\in\mathcal{D}_b}\left|\frac{F(K,L;q_0)}{F^{CD}(K,L;q_0)}-1\right|=0,$$

where the constant returns to scale Cobb-Douglas production function is given by

$$F^{CD}(K,L;q_0):=\mathcal{A}(q_0)^{-(1-\alpha)}L^{\alpha}K^{1-\alpha}, \tag{13}$$

and where $\mathcal{A}>0$ is a constant that only depends on parameters. Furthermore, the first derivatives of F converge in the analogous way to their F^{CD} counterparts.

In addition to establishing the convergence to Cobb-Douglas, this result shows that the long-run total factor productivity associated with the task-based technology is proportional to $q_0^{-(1-\alpha)}$. As capital becomes better and better at completing tasks, fewer resources are needed to complete one unit of output, and a given bundle of factors (K,L) produces more. This intuition explains why the impact of a movement in the frontier q_0 on TFP depends on the capital share $1-\alpha$.

3.9 On the impossibility of exact Cobb-Douglas representation

Proposition 2 shows that in the limit as $q_0\to 0$, the task-based technology with underlying Pareto distribution converges to the Cobb-Douglas production function. Away from that limit, however, the technology deviates from Cobb-Douglas. This raises the question: is there an alternative task distribution $\tilde{f}(q)$ that would yield Cobb-Douglas production at every point along the development path, not only in the limit as $q_0\to 0$. The following result provides an answer.

Proposition 3. *Consider the task-based production model with a capital requirement distribution characterized by a continuous probability density function $\tilde{f}(q)$ with support $[q_0, \infty)$, where $q_0 > 0$. There exists no distribution $\tilde{f}$ such that the associated production function $F(K, L)$ is of the Cobb-Douglas form for all factor endowments $(K, L) \in \mathbb{R}^2_{++}$.*

The proof of this proposition shows that a *necessary* condition for a Cobb-Douglas representation is that the underlying task distribution be Pareto. However, as we have seen, even that distribution does not yield Cobb-Douglas for $q_0 > 0$. It follows that no such distribution exists.

4 Growth Implications

We now embed the task-based technology in the neoclassical growth model of Section 2 and characterize the equilibrium dynamics. The economy's long-run behavior is standard: since the task-based technology converges to Cobb–Douglas (Proposition 2), the economy converges to a balanced growth path. Away from the long run, however, the task-based micro-foundations generate richer dynamics. Early in the development process, the technological frontier is too high for capital to be profitable, and the economy is trapped in a pre-industrial regime with no investment and no growth. As the frontier improves, the economy undergoes an endogenous industrial takeoff and transitions toward the balanced growth path. We characterize these dynamics in this section.

4.1 Long-run behavior

The behavior of the limiting Cobb-Douglas economy is standard. We denote the asymptotic growth rate of output per worker by

$$\nu := \frac{1-\alpha}{\alpha}\phi g,$$

which is determined by the rate of technological progress ϕg and the labor share parameter α. We use this quantity to define variables detrended per effective worker:

$$\hat{k}_t := \frac{K_t}{Le^{\nu t}}, \quad \text{and} \quad \hat{c}_t := \frac{C_t}{Le^{\nu t}}.$$

In the Cobb-Douglas model the detrended capital stock $\hat{k}_t$ converges monotonically to a unique steady state given by

$$\hat{k}^*_{CD} := \left(\frac{(1-\alpha)\mathcal{A}}{\delta+\rho+\nu}\right)^{\frac{1}{\alpha}}.$$

The following result establishes that the task-based economy converges to this same trajectory in the long run.

Proposition 4. *Suppose $K_0 > 0$. The capital per effective worker in the task-based economy $\hat{k}_t$ converges to the unique steady state of the Cobb-Douglas economy along the optimal equilibrium* *path:*

$$\lim_{t\to\infty} \hat{k}_t = \hat{k}^*_{CD}. \tag{14}$$

Furthermore, the asymptotic growth rates of K_t, C_t and Y_t converge to ν.

The model with task-based micro-foundations therefore preserves the desirable long-run properties of the neoclassical growth model, consistent with the empirical regularity of stable long-run growth.

4.2 Pre-industrial regime

A distinguishing feature of the task-based production function is that the marginal product of capital is bounded near zero. In the standard neoclassical model, the Inada condition $\lim_{K\to 0} F_K = \infty$ ensures that the return to the first unit of capital is arbitrarily high, so investment is always profitable. In the task-based economy, even the simplest task requires $q_0(t)$ units of capital per unit of output, so the return to the first unit of capital is finite. When this return falls short of the household's required rate of return, no investment occurs and the economy stagnates. To simplify the exposition, we assume that the economy begins with no capital ($K_0 = 0$).[7]

In this regime, all tasks are performed by labor. Since tasks are complements and each must be completed at least at intensity A, the firm sets $a(i) = A$ for all i and total labor demand is $L = \int_0^1 A\, di \cdot Y = AY$. The production function is therefore linear:

$$Y = A^{-1}L.$$

The wage equals the marginal product of labor, $w = A^{-1}$, and the capital stock is zero.

Now consider whether a firm would want to use a small amount of capital. The most productive use of a marginal unit would be to automate the simplest task in the economy—the one with the lowest capital requirement $q_0(t)$. Automating this task at intensity A requires $q_0(t)A$ units of capital but saves A units of labor, worth wA. The return per unit of capital invested is therefore

$$\bar{r} = \frac{wA}{q_0(t)A} = \frac{w}{q_0(t)} = \frac{1}{Aq_0(t)}.$$

The household, however, will only supply capital if the return covers the rate of time preference and depreciation, $\rho + \delta$.[8] Otherwise, it is better to consume any existing capital. Consequently, the economy remains trapped in the pre-industrial regime if the maximum return the firm can offer

[7] The analysis is similar when $K_0 > 0$, but in that case there is disinvestment during the pre-industrial regime.

[8] This is a consequence of the Euler equation $\dot{C}/C = r - \rho - \delta$. During the pre-industrial phase, production $Y = A^{-1}L$ is constant and there is no investment, so consumption is also constant ($\dot{C} = 0$). The required rate to begin investing, which implies $\dot{C} < 0$, is therefore $r = \rho + \delta$.

falls short of the household's required return:

$$\bar{r} < \rho + \delta \iff q_0\left(t\right) > \frac{1}{A\left(\rho + \delta\right)}. \tag{15}$$

Condition (15) defines the pre-industrial regime. Mathematically, it exists because of the lack of Inada condition for capital on the production function: $\lim_{K \to 0} F_K < \infty$. This bounded marginal product prevents the bootstrapping dynamic of standard growth models, where high returns at low capital levels guarantee the initial takeoff.

Escape from the trap is driven by the exogenous drift of the technological frontier: $q_0(t) = \exp(-\phi g t)$. As the capital requirement of the simplest tasks declines, the virtual marginal product $\bar{r}$ rises. At time

$$T_{\text{takeoff}} = \frac{1}{\phi g} \log\left(A\left(\rho + \delta\right)\right), \tag{16}$$

the frontier crosses the threshold in (15) and investment becomes profitable. A higher rate of technological progress ϕg accelerates the takeoff. Greater impatience ρ or faster depreciation δ raises the user cost of capital, requiring a lower frontier before the first machine is worth building. Economies that are otherwise identical but differ in ϕg will industrialize at different times, a prediction broadly consistent with the uneven timing of industrialization across countries. Figure 3 illustrates the pre-industrial regime and the takeoff.

4.3 Transition dynamics

Once the takeoff threshold is crossed, the economy begins a transition toward the balanced growth path. To characterize this transition, it is convenient to work with the *automation depth* $u_t := q_t^* / q_0\left(t\right) > 1$, which measures the extent of automation relative to the technological frontier. When $u_t = 1$, the economy is at the onset of automation. As $u_t \to \infty$, automation extends deep into the Pareto tail and the economy approaches the Cobb-Douglas limit.

Proposition 5 (Exact Transition Dynamics)**.** *For $t > T_{takeoff}$, the competitive equilibrium of the task-based economy is governed by*

$$\dot{K}_t = y\left(u_t\right) L - C_t - \delta K_t, \tag{17}$$

$$\frac{\dot{C}_t}{C_t} = r\left(u_t, q_0\left(t\right)\right) - \rho - \delta, \tag{18}$$

where $q_0\left(t\right) = e^{-\phi g t}$ and $u_t > 1$ is the unique solution to

$$\frac{K_t}{L} = \frac{\zeta\, q_0\left(t\right)}{\beta}\, u_t \left(1 - u_t^{-\beta}\right). \tag{19}$$

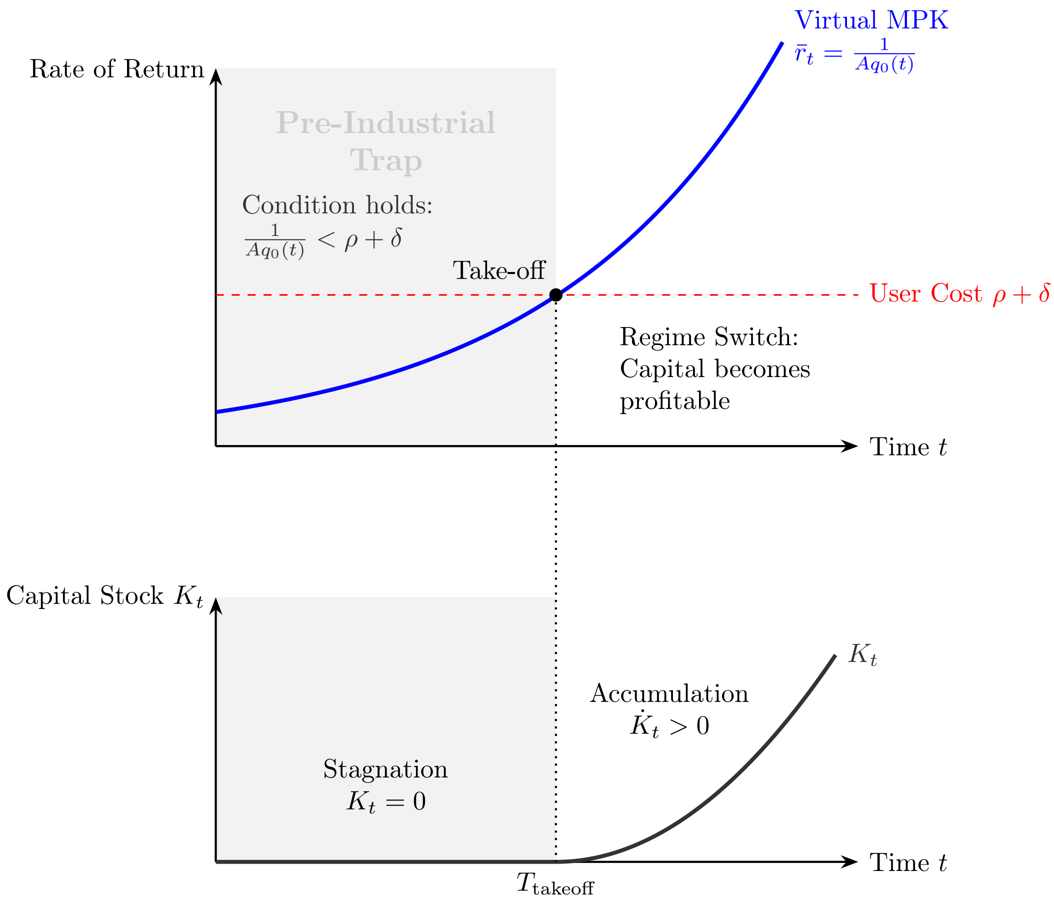

Figure 3: The pre-industrial trap and the beginning of the industrial era

Output per worker and the rental rate of capital are given by

$$y(u) := \frac{Y}{L} = \frac{1}{A} u^{\frac{\zeta}{1-\gamma}} \left[\frac{\beta}{(1-\gamma) - \zeta\, u^{-\beta}} \right]^{\frac{\gamma}{1-\gamma}}, \tag{20}$$

$$r(u, q_0) := F_K = \frac{1}{Aq_0} u^{-\alpha} \left[\frac{\beta}{(1-\gamma) - \zeta\, u^{-\beta}} \right]^{\frac{1}{1-\gamma}}. \tag{21}$$

Along the transition, the labor share is given by

$$s_L(u) = \frac{\beta}{(1-\gamma) - \zeta\, u^{-\beta}}. \tag{22}$$

The system (17)–(19) fully characterizes the equilibrium dynamics. The automation depth u_t is a sufficient statistic for the transition: all equilibrium quantities depend on time only through u_t (and, for the rental rate, the frontier $q_0(t)$). At each instant, the implicit equation (19) determines u_t from the current capital stock K_t and the exogenous frontier $q_0(t)$. Since the right-hand side of (19) is strictly increasing in u (for $u > 1$), the solution is unique. Given u_t, output, the rental rate, and all other equilibrium quantities follow from the explicit formulas above. The paths of capital K and consumption C are determined by (17)–(18).

The expressions reveal two monotonicity properties that hold throughout the transition. The labor share $s_L(u)$ is decreasing in u: it equals one at the onset of automation ($u = 1$), when all tasks are performed by labor, and declines smoothly toward the Cobb-Douglas limit α as $u \to \infty$. Output per worker $y(u)$ is increasing in u: as capital takes over tasks, workers are freed from routine activities and the economy produces more with the same labor force. Once $K_t > 0$, Proposition 4 ensures that the economy eventually converges to the balanced growth path.

Numerical illustration. Figures 4 and 5 illustrate these dynamics in the Leontief limit ($\gamma = 0$), comparing the task-based economy (blue) to the Cobb-Douglas benchmark (red).[9] The two economies share the same long-run destination but reach it in fundamentally different ways.

In the standard Cobb-Douglas economy, capital is scarce at $t = 0$ but its marginal product is high—infinitely so near zero, thanks to the Inada condition. Investment is therefore most profitable at the start and becomes less so over time as diminishing returns set in. The economy converges monotonically to the balanced growth path: fast accumulation early, gradual deceleration later (panel (a) of Figure 4). The growth rate of output per worker starts high and falls toward ν from above (panel (d)).

The task-based economy tells a very different story. During the pre-industrial phase ($t < T_{\text{takeoff}} \approx 30$ years in the figure), no capital is accumulated, output per worker is flat at $Y/L = 1/A$, and the labor share is one (panels (a)–(c) of Figure 4). Workers perform all tasks themselves, but productivity is low. After takeoff, capital accumulation begins, but slowly at first: the marginal product of capital is finite and only modestly above the household's required return. As the frontier $q_0(t)$ continues to decline, however, a reinforcing loop takes hold. Capital accumulation raises worker productivity and therefore wages. Higher wages raise the automation cutoff $q^* = w/r$, making additional tasks worth automating. The newly automated tasks further raise labor productivity, feeding back into yet more accumulation. As a result, capital accumulation *accelerates* rather than decelerates. Panel (d) of Figure 4 shows this clearly: the growth rate of output per worker rises from zero and converges to the balanced growth rate ν from below; the mirror image of the Cobb-Douglas economy. The two economies approach the same long-run growth rate from opposite directions.

The behavior of factor prices reinforces the contrast (Figure 5). In the Cobb-Douglas economy, the scarcity of capital at $t \approx 0$ depresses the marginal product of labor, so wages start near zero and rise as capital accumulates. Workers need machines to be productive. In the task-based economy, the opposite is true: wages are flat at $w = 1/A$ throughout the pre-industrial phase, because workers can perform every task themselves. Once automation begins, wages rise as capital takes over the

[9] The parameter values used in Figures 4 and 5 are: $\gamma = 0$, $\zeta = 0.4$, $A = 20$, $\rho = 0.04$, $\delta = 0.05$, $\phi = 0.5$, $g = 0.04$ and $L = 1$. These imply a long-run labor share of $\alpha = 1 - \zeta = 0.6$, consistent with the historical US average, and a balanced growth rate of output per worker of $\nu \approx 1.3\%$ per year.

routine tasks. As $t \to \infty$, the two economies converge to the same factor prices, but the early dynamics are strikingly different.

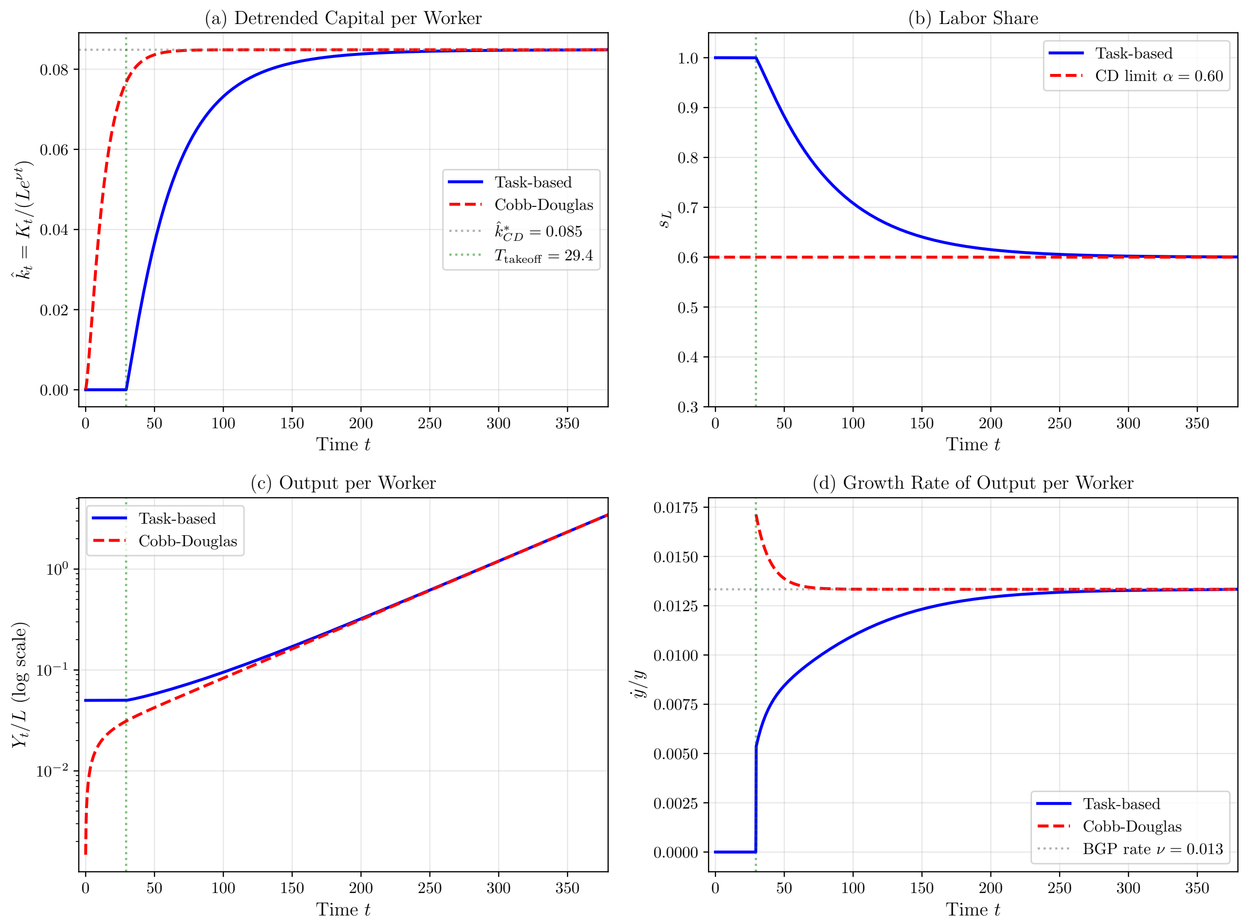

Figure 4: Transition dynamics: task-based vs Cobb-Douglas.

Note: The Cobb-Douglas curve in Panel (d) is truncated before T_{takeoff}.

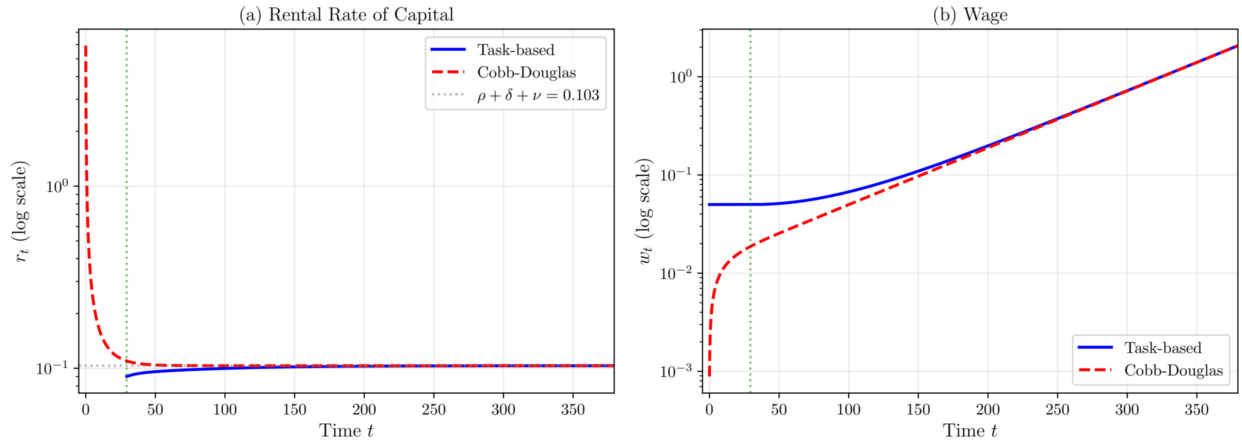

Figure 5: Factor prices: task-based vs Cobb-Douglas

5 Automation Probabilities and the Pareto Tail

One key primitive of the model is the Pareto distribution of task-level capital requirements, which determines the long-run labor share. In this section, we use data on the automation exposure of US occupations to assess this distributional assumption and estimate the tail index ζ. The exercise relies on a simple mapping: we interpret each occupation as performing a single representative task. Under this interpretation, an occupation's probability of being automated reflects how close its representative task is to the automation frontier. The Pareto tail then governs the cross-sectional distribution of these probabilities: occupations whose tasks are near the frontier have high automation probabilities, while those deep in the tail have low ones. We derive this prediction formally in Section 5.1, take it to data on occupation-level automation probabilities constructed by Frey and Osborne (2017), and show that the implied labor share falls in the historically observed range. Throughout this section, we work in the Leontief limit ($\gamma = 0$), so that the labor share is $\alpha = 1 - \zeta$.

5.1 From the model to observable automation probabilities

Consider the economy on a balanced growth path. Each task currently performed by labor has a capital requirement q that exceeds the automation cutoff $q^* = w/r$. Define the log-distance from the cutoff:

$$y = \log\left(\frac{q}{q^*}\right) > 0.$$

Since the stationary distribution of capital requirements among labor tasks is Pareto with tail index ζ, the log-distance y follows an exponential distribution:

$$y \sim \text{Exp}(\zeta), \qquad f(y) = \zeta e^{-\zeta y}, \quad y > 0.$$

A task with a small y is near the automation frontier: a modest improvement in technology would push its capital requirement below the cutoff. A task deep in the tail (large y) is far from being automated.

Now consider what happens over a horizon of T years. Each task's capital requirement evolves according to the geometric Brownian motion $d\log q = -g\,dt + \sigma\,dW$, and the cutoff q^* grows at rate ν (the balanced growth rate of real wages). A task currently at log-distance y from the cutoff becomes automated within T years with probability

$$p(y) \approx \Phi\left(\frac{(g+\nu)\,T - y}{\sigma\sqrt{T}}\right), \tag{23}$$

where Φ is the standard normal cumulative distribution function.[10] Tasks close to the cutoff (small y) have high automation probability; tasks deep in the tail have low probability.

5.2 Data and calibration

The data come from Frey and Osborne (2017), who use a machine learning classifier trained on expert assessments to assign each of 702 US occupations a probability $p_j \in (0,1)$ of computerization over the next one to two decades. Occupations such as telemarketers and tax preparers receive high probabilities; surgeons and elementary school teachers receive low ones. Under the representative-task interpretation, the Frey-Osborne probability p_j maps directly to $p(y_j)$ in equation (23).

We weight each occupation by its employment level from the Bureau of Labor Statistics Occupational Employment and Wage Statistics survey (May 2016 release). After excluding occupations with extreme probabilities ($p = 0$ or $p = 1$), 697 occupations remain.

The automation probability in (23) involves four parameters (ζ, σ, g, ν) in addition to the horizon T. On the balanced growth path, these satisfy the constraint

$$g + \nu = \frac{\nu}{\zeta} + \frac{1}{2}\zeta\sigma^2, \tag{24}$$

which follows from combining the stationary condition for the Pareto tail ($\zeta = 2g(1-\phi)/\sigma^2$; see Assumption 2) with the balanced growth relationship $\nu = \zeta\phi g/(1-\zeta)$.[11] This constraint reduces the free parameters to three. We calibrate two externally and estimate the third from the data.

The balanced growth rate of real wages, $\nu \approx 1.5\%$ per year, is taken from BLS data on real hourly compensation over 1950–2019. The automation horizon $T = 20$ years corresponds to the upper end of the range described by Frey and Osborne (2017) ("over the next decade or two").

[10] Equation (23) is the standard normal approximation to the first-passage probability for Brownian motion with drift, omitting a reflection term that is negligible for the parameter values considered here. The exact formula is provided in Appendix A.

[11] On the balanced growth path, real wages grow at rate ν and the interest rate is constant, so $q^* = w/r$ grows at rate ν. The frontier q_0 declines at rate ϕg. The balanced growth rate of output per worker is $\nu = (1-\alpha)\phi g/\alpha = \zeta\phi g/(1-\zeta)$.

The innovation drift g is the average rate at which the capital requirement of a given task falls. At $g = 5\%$, the cost of automating a specific task halves roughly every 14 years. A natural benchmark is the quality-adjusted decline in equipment prices: Gordon (2007) and Cummins and Violante (2002) estimate this rate at approximately 3–5% per year over the postwar period, so our baseline of $g = 5\%$ is at the upper end of their range. We explore sensitivity to g below.

Given (g, ν, T), the constraint (24) reduces the model to a single free parameter (ζ), which can be estimated using maximum likelihood from the automation probability data.

5.3 Results

At the baseline calibration ($g = 5\%$, $\nu = 1.5\%$, $T = 20$), the estimation yields

$$\hat{\zeta} = 0.41, \qquad \hat{\alpha} = 1 - \hat{\zeta} = 0.59,$$

with a standard error of 0.014 on both $\hat{\zeta}$ and $\hat{\alpha}$ (95% confidence interval for $\hat{\alpha}$: $[0.56, 0.62]$). The implied volatility is $\hat{\sigma} = 0.37$ and the frontier share $\hat{\phi} = 0.43$. The implied labor share $\hat{\alpha} = 0.59$ falls squarely in the range of 0.57–0.65 observed in the US postwar data. It is reassuring that the micro-level automation probability data, once interpreted through the lens of the model, lead to a macro quantity that is consistent with aggregate data.

Model fit. Figure 6 compares the empirical cumulative distribution function of automation probabilities (employment-weighted) with the model-predicted CDF at the baseline estimates. Both distributions are restricted to labor tasks ($p < p_{\max} = 0.78$). The model captures the broad shape of the cross-sectional distribution of automation exposure but exhibits systematic deviations: it underpredicts the mass of occupations with very low automation probabilities and overpredicts the mass at intermediate values. These departures may reflect heterogeneity in the tasks bundled within each occupation, deviations from the exact exponential form in the deep tail, or both.

Sensitivity to g. Figure 7 displays $\hat{\zeta}$ and $\hat{\alpha}$ as functions of g for the baseline ν and T. The horizontal band marks the US historical labor share range $\alpha \in [0.57, 0.65]$. The curve crosses this band for $g \in [4.5\%, 6\%]$, which corresponds to automation half-lives of 12–15 years, within the range of estimates in the quality-adjusted equipment price literature. We report additional robustness results, varying both g and T, in Appendix A.

5.4 Discussion

The estimates also provide a micro-founded interpretation of aggregate TFP growth. In the Cobb-Douglas limit, TFP takes the form $(q_0)^{-(1-\alpha)}$ (up to a constant): productivity growth arises entirely from the expanding frontier of automatable tasks, rather than from disembodied technical

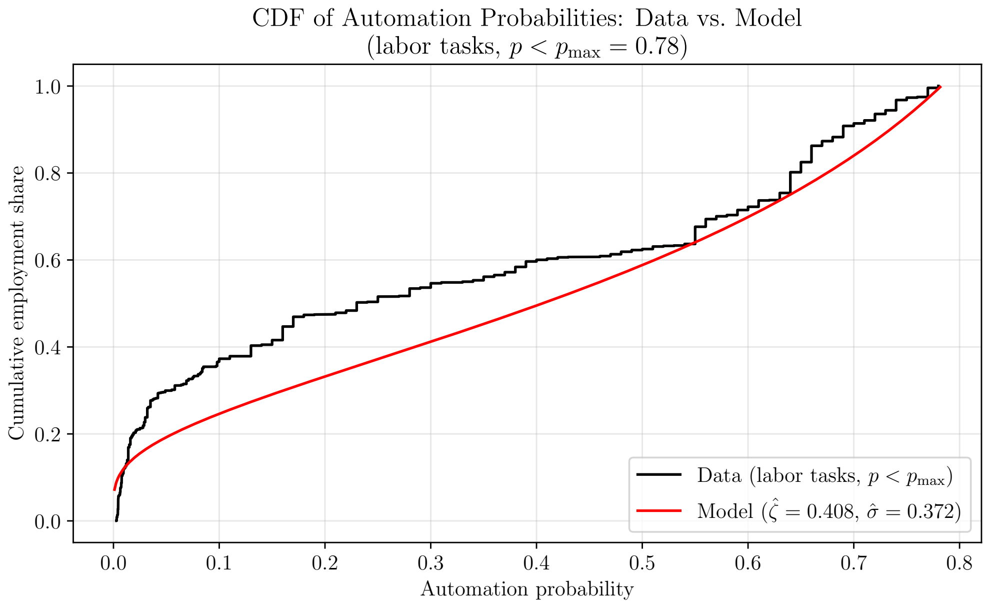

Figure 6: Cumulative distribution of automation probabilities: data (employment-weighted) versus model prediction. Both distributions are conditional on labor tasks ($p < p_{\max} = 0.78$). The model is evaluated at the baseline estimates $\hat{\zeta} = 0.41$, $\hat{\sigma} = 0.37$.

change. With our estimates, the implied TFP growth rate is $(1 - \hat{\alpha})\,\hat{\phi}g = 0.41 \times 0.022 \approx 0.9\%$ per year, in the range of standard estimates for the US postwar period. This is a further consistency check: the tail index estimated from micro-level automation probabilities, combined with the calibrated pace of innovation, produces a TFP growth rate that aligns with aggregate data.

In summary, the Frey-Osborne automation probabilities provide a micro-level test of the model's core distributional assumption. The cross-section of automation exposure is broadly consistent with the exponential distribution of log-distances implied by the Pareto tail. In addition, for innovation rates in the range documented by the quality-adjusted equipment price literature, the model delivers a labor share and a TFP growth rate that align with US aggregate data. This concordance between independently measured micro and macro moments is encouraging, particularly for a model that was not calibrated to match these moments.

While these results are reassuring, we interpret them cautiously. The model is stylized, our assumption that one task corresponds to one occupation is crude, and the Frey and Osborne (2017) probabilities, though a natural starting point, have well-known limitations.[12] More recent measures of automation exposure, such as the patent-based indices of Webb (2019) or the AI-specific assessments of Eloundou et al. (2024), could be used to update the exercise and, in particular, to test whether the Pareto tail is thinning in response to advances in artificial intelligence.

[12] The original expert assessments covered a small subset of occupations and were extrapolated to the full set using a machine learning classifier; the automation horizon ("one to two decades") is imprecise; and the probabilities predate the recent wave of large language models, which has shifted the frontier of automatable tasks toward cognitive and creative work.

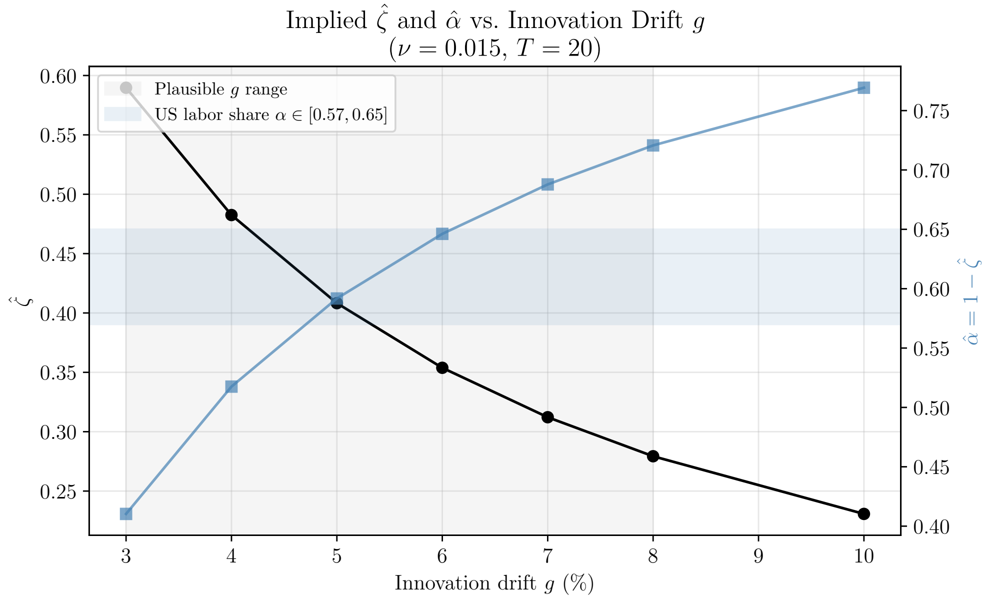

Figure 7: Implied $\hat{\zeta}$ and $\hat{\alpha}$ as functions of the innovation drift g, with $\nu = 1.5\%$ and $T = 20$ years. The horizontal band shows the US historical labor share range $\alpha \in [0.57, 0.65]$. The curve crosses this band for $g \in [4.5\%, 6\%]$, corresponding to task-level automation half-lives of 12–15 years.

6 Concluding Remarks

This paper develops a task-based model of production in which capital-augmenting technological progress gives rise to a Pareto distribution of task-level capital requirements. Despite ongoing automation, the labor share converges to a positive constant and the production technology converges to Cobb–Douglas. Embedded in a neoclassical growth model, the economy exhibits a preindustrial trap, an endogenous industrial takeoff, and convergence to a balanced growth path. Using occupation-level automation probabilities, we show that the cross-sectional pattern of automation exposure is consistent with the model's distributional predictions and implies a labor share in the historically observed range.

A central insight of the paper is that labor share stability follows from a statistical regularity (the power-law distribution of capital requirements) rather than from parametric restrictions on technology or preferences. This regularity emerges naturally from the proportional, scale-free nature of the innovation process, consistent with the ubiquity of power laws in economic and natural phenomena. However, statistical regularities are not immutable. As we discuss in Section 3.5, changes in the structural properties of the innovation process, such as those plausibly associated with the ICT revolution or the emergence of artificial intelligence, can alter the tail of the distribution and shift the labor share to a new level. The model therefore provides a unified framework for understanding both the long historical stability of the labor share and its more recent decline.

Several directions for future research seem promising. First, the innovation process in our model is exogenous. Endogenizing the direction of technical change would shed light on whether market forces tend to preserve or erode the power-law tail. Second, our framework features a single type of labor. Introducing heterogeneous workers with different comparative advantages across tasks would allow the model to speak to the distributional consequences of automation, including wage inequality and job polarization. Third, the tail parameter ζ is the key determinant of the labor share, but we have not estimated it. Bringing the model to data, using information on the cross-sectional distribution of capital intensity across tasks or occupations, could discipline the theory and sharpen its predictions.

Several directions for future research seem promising. First, the innovation process in our model is exogenous. Endogenizing the direction of technical change would shed light on whether market forces tend to preserve or erode the power-law tail. Second, our framework features a single type of labor. Introducing heterogeneous workers with different comparative advantages across tasks would allow the model to speak to the distributional consequences of automation, including wage inequality and job polarization. Third, our empirical exercise interprets each occupation as performing a single representative task. Since occupations in practice bundle activities at heterogeneous distances from the automation cutoff, developing a richer occupation-to-task mapping could improve the fit and sharpen the estimates of the tail parameter.

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Online Appendix (Not for Publication)

A Estimation Details

This appendix provides the derivation of the likelihood function and the closed-form estimator for the tail index ζ used in Section 5. We establish a non-identification result—that automation probability data identify only the composite $c = \zeta\sigma$, not ζ and σ separately—which explains why the calibration of g is needed to recover ζ. We then derive the mapping from the estimated composite $\hat{c}$ to the tail index $\hat{\zeta}$, and present standard errors, robustness results, and a formal goodness-of-fit test.

A.1 Likelihood function

Let p_j denote the automation probability for occupation j and e_j its employment. Define $x_j = \Phi^{-1}(p_j)$. Given parameters (ζ, σ) and calibrated (g, ν, T), the implied log-distance from the automation cutoff is

$$\hat{y}_j = (g + \nu)T - \sigma\sqrt{T} \cdot x_j.$$

Occupations with $\hat{y}_j > 0$ are classified as labor tasks; those with $\hat{y}_j \leq 0$ are classified as already automated.

This inversion uses the normal approximation to the first-passage probability. The exact first-passage probability for a Brownian motion with drift $\mu = g + \nu$ and volatility σ is

$$p(y) = \Phi\left(\frac{\mu T - y}{\sigma\sqrt{T}}\right) + e^{-2\mu y/\sigma^2}\Phi\left(\frac{-\mu T - y}{\sigma\sqrt{T}}\right).$$

The second (reflection) term is negligible for $y > 0$ and $\mu > 0$: at the baseline estimates ($\mu = 0.065$, $\sigma = 0.37$, $T = 20$), the reflection term is below 10^{-3} for all labor tasks. We therefore use the normal approximation throughout. Importantly, the exact formula exhibits the same Brownian scaling invariance as the approximation, so the non-identification result established below would hold even with the exact formula.

The density of the observed probability p_j for a labor task is obtained by change of variables from the exponential density of y:

$$f_P(p) = \zeta\, e^{-\zeta y(p)} \cdot \frac{\sigma\sqrt{T}}{\varphi(\Phi^{-1}(p))},$$

where $\varphi(\cdot)$ is the standard normal density. The employment-weighted log-likelihood over labor tasks is

$$\ell(\zeta, \sigma) = \sum_{j:\hat{y}_j > 0} \tilde{w}_j \left[\log\zeta - \zeta\hat{y}_j + \log\sigma + \tfrac{1}{2}x_j^2 + \tfrac{1}{2}\log(2\pi)\right],$$

where $\tilde{w}_j = e_j / \sum_{k:\hat{y}_k>0} e_k$ are renormalized employment weights. Since the last two terms do not depend on (ζ, σ), maximizing the likelihood is equivalent to maximizing

$$\ell(\zeta, \sigma) = \log \zeta - \zeta \bar{y}_w + \log \sigma,$$

where $\bar{y}_w = \sum_{j:\hat{y}_j>0} \tilde{w}_j \hat{y}_j$ is the employment-weighted mean of the implied log-distances.

A.2 The Brownian scaling property and non-identification

Proposition 6. *The log-likelihood ℓ depends on (ζ, σ) only through the product $c = \zeta\sigma$. It does not depend on g.*

Proof. From the balanced growth path relationships (Section 2), ζ and σ can be expressed as functions of c, g, and ν: $\zeta = (\nu + c^2/2)/(g + \nu)$ and $\sigma = c(g + \nu)/(\nu + c^2/2)$. Substituting into each term:
Term 1: $\log \zeta + \log \sigma = \log\left[\frac{\nu + c^2/2}{g+\nu}\right] + \log\left[\frac{c(g+\nu)}{\nu+c^2/2}\right] = \log c$.
Term 2: The implied log-distance is

$$\hat{y}_j = (g + \nu)T - \frac{c(g+\nu)}{\nu + c^2/2}\sqrt{T} \cdot x_j,$$

so

$$\zeta \hat{y}_j = \frac{\nu + c^2/2}{g + \nu} \cdot \left[(g+\nu)T - \frac{c(g+\nu)}{\nu + c^2/2}\sqrt{T} \cdot x_j\right] = (\nu + c^2/2)T - c\sqrt{T} \cdot x_j.$$

Averaging: $\zeta \bar{y}_w = (\nu + c^2/2)T - c\sqrt{T} \cdot \bar{x}_w$.
Term 3: The cutoff for labor tasks, $\hat{y}_j > 0$, is equivalent to $x_j < \sqrt{T}(\nu + c^2/2)/c$, which depends only on c, ν, and T.

Combining: $\ell(c) = \log c - (\nu + c^2/2)T + c\sqrt{T} \cdot \bar{x}_w$, which involves ν and T but not g. □

This invariance is a consequence of the spatial scaling property of Brownian motion. The rescaling $(\zeta, \sigma) \to (\zeta/k, k\sigma)$ multiplies all log-distances y by k while leaving all first-passage probabilities unchanged, because the drift and volatility rescale proportionally. No amount of data on automation probabilities—regardless of sample size, precision, or additional time horizons—can break this invariance. Intuitively, a world with a thin Pareto tail (low ζ) and high volatility (σ) generates the same cross-section of automation probabilities as a world with a thick tail (high ζ) and low volatility.

The composite $\hat{c}$ is therefore the only quantity that the Frey-Osborne data can estimate. Recovering $\hat{\zeta}$ from $\hat{c}$ requires external information: the calibrated g breaks the invariance by fixing the absolute scale of the drift, which determines how much of the composite $c = \zeta\sigma$ is attributed to ζ versus σ through equation (26). Note also that $\hat{c}$ does not depend on g—the calibrated innovation rate enters only through the formula mapping $\hat{c}$ to $\hat{\zeta}$, not through the estimation itself.

A.3 Closed-form estimator

The first-order condition $\partial \ell / \partial c = 0$ yields

$$\frac{1}{c} - cT + \sqrt{T} \cdot \bar{x}_w = 0,$$

a quadratic in c with solution

$$\hat{c} = \frac{\bar{x}_w + \sqrt{\bar{x}_w^2 + 4}}{2\sqrt{T}}, \tag{25}$$

where $\bar{x}_w = \sum_{j:\hat{y}_j > 0} \tilde{w}_j \Phi^{-1}(p_j)$ is the employment-weighted mean of $\Phi^{-1}(p_j)$ over labor tasks. Since the set of labor tasks depends on $p_{\max} = \Phi(\sqrt{T}(\nu + \hat{c}^2/2)/\hat{c})$, which itself depends on $\hat{c}$, the estimator is computed iteratively: starting from all occupations, compute $\hat{c}$ from (25), determine $p_{\max}$, drop occupations with $p_j \geq p_{\max}$, recompute $\bar{x}_w$ and $\hat{c}$, and repeat until convergence. In our data, convergence occurs in two iterations.

Given the estimated $\hat{c}$ and the calibrated (g, ν), the tail index and remaining parameters follow from the balanced growth path relationships:

$$\hat{\zeta} = \frac{\nu + \hat{c}^2/2}{g + \nu}, \tag{26}$$

with implied labor share $\hat{\alpha} = 1 - \hat{\zeta}$, volatility $\hat{\sigma} = \hat{c}/\hat{\zeta}$, frontier drift $\hat{\phi g} = \nu(1 - \hat{\zeta})/\hat{\zeta}$, and frontier share $\hat{\phi} = \hat{\phi g}/g$.

A.4 Standard errors

The second derivative of the log-likelihood at the optimum is $\partial^2 \ell / \partial c^2 = -1/c^2 - T$. Treating each of the n labor-task occupations as an independent observation, the variance of $\hat{c}$ is

$$\mathrm{SE}(\hat{c}) = \frac{c}{\sqrt{n(1 + c^2 T)}}.$$

The standard error for $\hat{\zeta}$ follows by the delta method. From (26), $\partial \zeta / \partial c = c/(g + \nu)$, so

$$\mathrm{SE}(\hat{\zeta}) = \frac{c}{g + \nu} \cdot \mathrm{SE}(\hat{c}).$$

Since $\alpha = 1 - \zeta$, $\mathrm{SE}(\hat{\alpha}) = \mathrm{SE}(\hat{\zeta})$. These standard errors reflect sampling uncertainty in $\hat{c}$ conditional on the calibrated (g, ν, T). They do not capture uncertainty in the calibration of g; the sensitivity of $\hat{\zeta}$ to g is shown in Figure 7 and Figure 8.

A.5 Robustness

The composite $\hat{c}$ depends on the automation horizon T (through the transformation from probabilities to log-distances) but not on the innovation drift g (by the scaling invariance established above). The implied $\hat{\zeta}$ depends on both g and T through equation (26).

Figure 8 reports $\hat{\zeta}$ for a grid of (g, T) values. The columns show that $\hat{c}$ is re-estimated for each T: longer horizons compress the scale of log-distances, reducing $\hat{c}$ and therefore $\hat{\zeta}$ for any given g. Within each column, the variation across rows is purely mechanical (the formula (26) with fixed $\hat{c}$ and ν). The cells producing implied labor shares in the range $\alpha \in [0.57, 0.65]$ trace a diagonal through the table, running from $(g = 5\%, T = 20)$ to $(g = 7\%, T = 15)$.

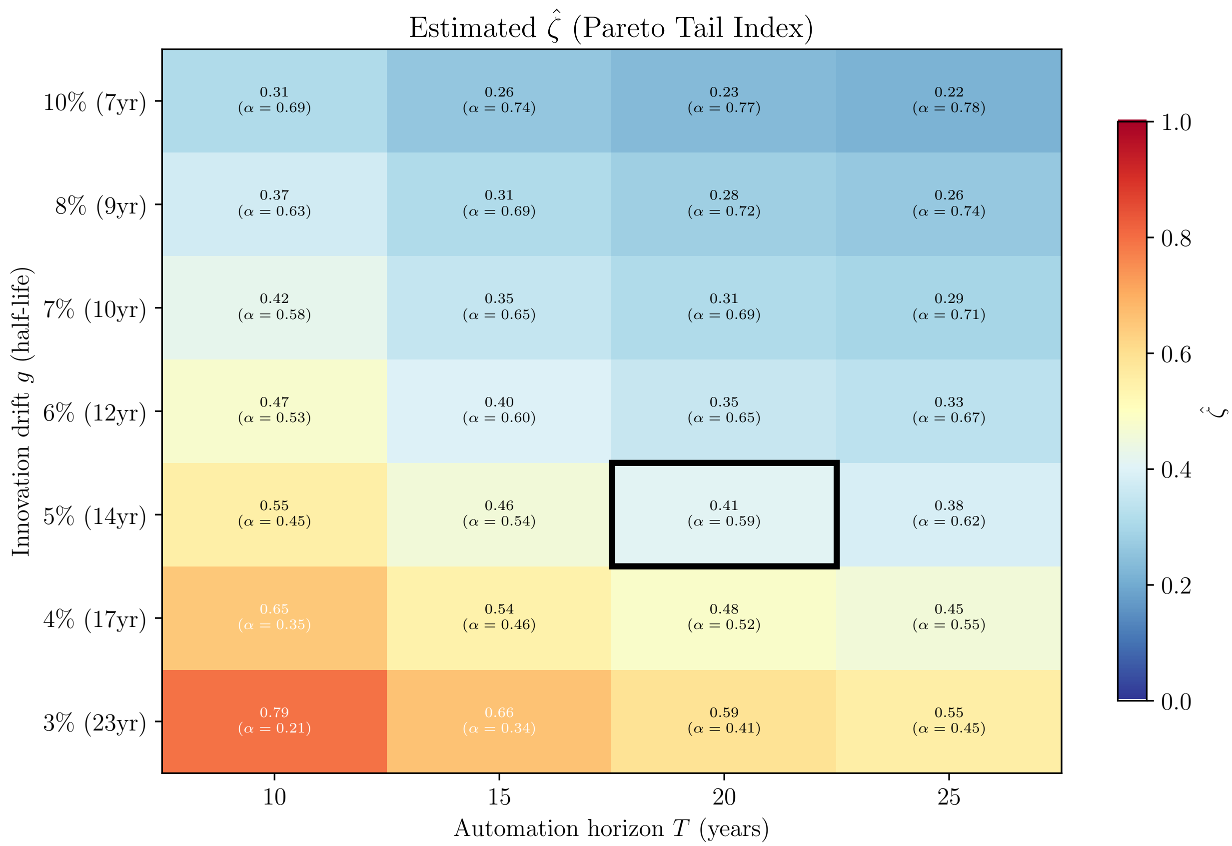

Figure 8: Estimated $\hat{\zeta}$ over a grid of innovation drift g and automation horizon T, with $\nu = 1.5\%$. The baseline cell $(g = 5\%, T = 20)$ is marked. Each cell reports $\hat{\zeta}$ and the implied labor share $\hat{\alpha} = 1 - \hat{\zeta}$.

B Omitted Proofs

This section contains the proofs that are not in the main text.

B.1 Proof of Lemma 1

Lemma 1. *Let $x_t := q_t e^{\phi g t}$ denote the capital requirement normalized by the technological frontier. The variable x_t admits a stationary Pareto distribution with probability density*

$$p(x) = \zeta x^{-\zeta-1}, \quad x \geq 1,$$

where the tail index ζ is given by

$$\zeta := \frac{g(1-\phi) + \sqrt{g^2(1-\phi)^2 + 2\sigma^2\chi}}{\sigma^2} \xrightarrow[\chi \to 0]{} 2g\frac{1-\phi}{\sigma^2}.$$

Equivalently, the unnormalized capital requirement q_t evolves according to a process with a moving support $q_t \geq q_0(t)$ and probability density

$$f_t^*(q) := \zeta q_0(t)^{\zeta} q^{-\zeta-1}, \quad q \geq q_0(t) := e^{-\phi g t}.$$

Proof. Define the transformed variable $x_t := q_t e^{\phi g t}$. Since $q_t \geq e^{-\phi g t}$, we have $x_t \in [1, \infty)$. Let $y_t := \log x_t = \log q_t + \phi g t$. Using Ito calculus and the process for $\log q_t$, the evolution of y_t is

$$dy_t = d(\log q_t) + \phi g dt = (-g dt + \sigma dW_t) + \phi g dt = \mu dt + \sigma dW_t, \tag{27}$$

where the drift is $\mu := -g(1-\phi)$. Since $\phi < 1$ and $g > 0$, we have $\mu < 0$. The process y_t is defined on $[0, \infty)$ with a reflecting barrier and reinjection at $y = 0$.

Let $p(y)$ denote the stationary probability density function of y. The probability flux $J(y)$ is given by

$$J(y) := \mu p(y) - \frac{\sigma^2}{2} p'(y). \tag{28}$$

In the stationary state, the net change in probability mass must be zero. The divergence of the flux must balance the death rate χ. This yields the Ordinary Differential Equation (ODE):

$$0 = -\frac{dJ}{dy} - \chi p(y) \implies \frac{\sigma^2}{2} p''(y) - \mu p'(y) - \chi p(y) = 0. \tag{29}$$

We seek a solution of the form $p(y) = Ce^{ry}$. Substituting this into the ODE yields the characteristic equation

$$\frac{\sigma^2}{2} r^2 - \mu r - \chi = 0. \tag{30}$$

The roots are $r = \frac{\mu \pm \sqrt{\mu^2 + 2\sigma^2\chi}}{\sigma^2}$. Since $\chi \geq 0$, the discriminant is larger than μ^2, implying two real roots: one positive (r_1) and one negative (r_2). For $p(y)$ to be integrable on $[0, \infty)$, we must select the negative root r_2. We define $\zeta := -r_2$:

$$\zeta = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2\chi}}{\sigma^2} = \frac{g(1-\phi) + \sqrt{g^2(1-\phi)^2 + 2\sigma^2\chi}}{\sigma^2}. \tag{31}$$

Thus, the density is $p(y) = \zeta e^{-\zeta y}$ (where $C = \zeta$ ensures the integral sums to 1).

To verify the boundary condition, note that the total death rate is $\int_0^\infty \chi p(y)dy = \chi$. Stationarity requires the influx of "reborn" tasks at the boundary $y = 0$ to equal the total death rate. The flux at the boundary is:

$$J(0) = \mu p(0) - \frac{\sigma^2}{2} p'(0) = \zeta\left(\mu + \frac{\sigma^2}{2}\zeta\right). \tag{32}$$

From the characteristic equation, $\frac{\sigma^2}{2}\zeta^2 + \mu\zeta = \chi$. Thus $J(0) = \chi$, satisfying the condition.

Finally, we transform back to q. The CDF of $x = e^y$ is:

$$F_X(x) = \Pr(e^Y \leq x) = \Pr(Y \leq \log x) = \int_0^{\log x} \zeta e^{-\zeta u} du = 1 - x^{-\zeta}. \tag{33}$$

Differentiating yields the PDF $p_X(x) = \zeta x^{-\zeta-1}$ for $x \geq 1$. Using the definition $q = xq_0(t)$ where $q_0(t) = e^{-\phi g t}$, the distribution of q is:

$$f_t^*(q) = p_X\left(\frac{q}{q_0(t)}\right)\frac{1}{q_0(t)} = \zeta\left(\frac{q}{q_0(t)}\right)^{-\zeta-1}\frac{1}{q_0(t)} = \zeta q_0(t)^\zeta q^{-\zeta-1}, \tag{34}$$

for $q \geq q_0(t)$, which matches the statement in the Lemma. □

B.2 Proof of Lemma 2

Lemma 2. *Cost minimization implies a cutoff rule $q^* = w/r$. The optimal task intensity satisfies:*

$$a(q) = \begin{cases} a(q^*)\left(\frac{q}{q^*}\right)^{-\gamma} & \text{for } q < q^* \quad \text{(capital tasks)} \\ a(q^*) & \text{for } q \geq q^* \quad \text{(labor tasks)} \end{cases} \tag{7}$$

where $a(q^*)$ is the intensity at the cutoff, which is given by

$$a(q^*) = A\left(\frac{1-\gamma}{1-\gamma-\zeta}\left(\frac{q_0}{q^*}\right)^\zeta - \frac{\zeta}{1-\gamma-\zeta}\left(\frac{q_0}{q^*}\right)^{1-\gamma}\right)^{\frac{\gamma}{1-\gamma}}.$$

Proof. The firm minimizes the total cost of completing tasks to produce 1 unit of output. Let $p(i) = \min\{w, rq(i)\}$ be the cost of one unit of intensity allocated to task i. The firm solves

$$\min_{\{a(i)\}} \int p(i)a(i)\, di,$$

subject to

$$\left(\int a(i)^{\frac{\gamma-1}{\gamma}}\,di\right)^{\frac{\gamma}{\gamma-1}} \geq A.$$

The first-order condition with respect to $a\left(j\right)$ is

$$a\left(j\right)=\left(\frac{p\left(j\right)}{\lambda}\right)^{-\gamma}A.$$

where λ is the Lagrange multiplier on the constraint. Plugging back that expression in the binding constraint yields

$$\lambda=\left(\int\left(p\left(i\right)\right)^{1-\gamma}\,di\right)^{\frac{1}{1-\gamma}}.$$

For tasks assigned to labor, we have $p\left(i\right)=w$ and so

$$a\left(i\right)=\left(\frac{w}{\lambda}\right)^{-\gamma}A:=a\left(q^{*}\right).$$

For tasks assigned to capital, we have instead $p\left(i\right)=rq$ and so

$$a\left(j\right)=\left(\frac{rq\left(j\right)}{\lambda}\right)^{-\gamma}A.$$

Taking the ratio of these last two expressions yields (7).

To compute $a\left(q^{*}\right)$ in terms of fundamentals, we can go back to the constraint

$$A\leq\left(\int a(i)^{\frac{\gamma-1}{\gamma}}\,di\right)^{\frac{\gamma}{\gamma-1}}=\left(\int_{q_0}^{q^{*}}a\left(q\right)^{\frac{\gamma-1}{\gamma}}\,f_t^{*}\left(q\right)\,dq+\int_{q^{*}}^{\infty}a\left(q^{*}\right)^{\frac{\gamma-1}{\gamma}}\,f_t^{*}\left(q\right)\,dq\right)^{\frac{\gamma}{\gamma-1}}.$$

Using (5) and solving the labor integral first yields

$$\int_{q^{*}}^{\infty}a\left(q^{*}\right)^{\frac{\gamma-1}{\gamma}}\,\zeta q_0^{\zeta}q^{-\zeta-1}dq=a\left(q^{*}\right)^{\frac{\gamma-1}{\gamma}}\,\zeta q_0^{\zeta}\left[\frac{q^{-\zeta}}{-\zeta}\right]_{q^{*}}^{\infty}=a\left(q^{*}\right)^{\frac{\gamma-1}{\gamma}}\left(\frac{q_0}{q^{*}}\right)^{\zeta}.$$

For the capital integral, we get

$$\int_{q_0}^{q^{*}}a\left(q^{*}\right)^{\frac{\gamma-1}{\gamma}}\left(\frac{q}{q^{*}}\right)^{1-\gamma}\zeta q_0^{\zeta}q^{-\zeta-1}dq=a\left(q^{*}\right)^{\frac{\gamma-1}{\gamma}}\left(q^{*}\right)^{-(1-\gamma)}\zeta q_0^{\zeta}\int_{q_0}^{q^{*}}q^{-\gamma-\zeta}dq.$$

Since $1-\gamma-\zeta>0$ by condition (6), the integral becomes

$$\int_{q_0}^{q^{*}}a\left(q\right)^{\frac{\gamma-1}{\gamma}}\,f_t^{*}\left(q\right)\,dq=a\left(q^{*}\right)^{\frac{\gamma-1}{\gamma}}\,\frac{\zeta}{1-\gamma-\zeta}\left[\left(\frac{q_0}{q^{*}}\right)^{\zeta}-\left(\frac{q_0}{q^{*}}\right)^{1-\gamma}\right].$$

Combining the labor and capital integral, we get

$$A=\left(a\left(q^{*}\right)^{\frac{\gamma-1}{\gamma}}\frac{\zeta}{1-\gamma-\zeta}\left[\left(\frac{q_0}{q^{*}}\right)^{\zeta}-\left(\frac{q_0}{q^{*}}\right)^{1-\gamma}\right]+a\left(q^{*}\right)^{\frac{\gamma-1}{\gamma}}\left(\frac{q_0}{q^{*}}\right)^{\zeta}\right)^{\frac{\gamma}{\gamma-1}}.$$

Solving for $a\left(q^{*}\right)$, we find

$$a\left(q^{*}\right)=A\left(\frac{1}{1-\gamma-\zeta}\right)^{\frac{\gamma}{1-\gamma}}\left((1-\gamma)\left(\frac{q_0}{q^{*}}\right)^{\zeta}-\zeta\left(\frac{q_0}{q^{*}}\right)^{1-\gamma}\right)^{\frac{\gamma}{1-\gamma}}.$$

□

B.3 Proof of Lemma 3

Lemma 3. *Per unit of output, the firm's optimal demand for labor L is given by*

$$\frac{L}{Y}=\underbrace{a\left(q^{*}\right)}_{\substack{\textit{labor requirement of}\\ \textit{labor tasks}}}\times\underbrace{\left(\frac{q_0}{q^{*}}\right)^{\zeta}}_{\substack{\textit{mass of}\\ \textit{labor tasks}}},\tag{8}$$

and its optimal demand for capital K is given by

$$\frac{K}{Y}=\underbrace{q^{*}a\left(q^{*}\right)}_{\substack{\textit{marginal capital}\\ \textit{usage}}}\times\underbrace{\frac{\zeta}{1-\gamma-\zeta}\left(\frac{q_0}{q^{*}}\right)^{\zeta}D\left(\frac{q_0}{q^{*}}\right)}_{\substack{\textit{effective measure of}\\ \textit{automated tasks}}},\tag{9}$$

where

$$D\left(\frac{q_0}{q^{*}}\right):=1-\left(\frac{q_0}{q^{*}}\right)^{1-\gamma-\zeta},$$

is the effective automation depth.

Proof. The aggregate demand for labor is the integral of labor intensity over the set of tasks performed by labor ($q\geq q^{*}$):

$$\frac{L}{Y}=\int_{q^{*}}^{\infty}a(q)f_t^{*}(q)dq.$$

Substituting the constant intensity $a(q)=a(q^{*})$ and the PDF:

$$\frac{L}{Y}=\int_{q^{*}}^{\infty}a(q^{*})\zeta q_0^{\zeta}q^{-\zeta-1}dq=a(q^{*})\zeta q_0^{\zeta}\left[\frac{q^{-\zeta}}{-\zeta}\right]_{q^{*}}^{\infty}=a(q^{*})q_0^{\zeta}(q^{*})^{-\zeta}.$$

Rearranging terms yields the labor expression.

The aggregate demand for capital is the integral of capital intensity weighted by complexity q

over the set of automated tasks ($q_0 \leq q < q^*$):

$$\frac{K}{Y} = \int_{q_0}^{q^*} qa\left(q\right) f_t^*\left(q\right) dq.$$

Substituting the policy $a\left(q\right) = a\left(q^*\right) \left(q/q^*\right)^{-\gamma}$ and the PDF:

$$\frac{K}{Y} = \int_{q_0}^{q^*} q \left[a\left(q^*\right) \left(q^*\right)^{\gamma} q^{-\gamma}\right] \left[\zeta q_0^{\zeta} q^{-\zeta-1}\right] dq.$$

Evaluating the integral, we find

$$\frac{K}{Y} = a\left(q^*\right) \left(q^*\right)^{\gamma} \zeta q_0^{\zeta} \left[\frac{q^{1-\gamma-\zeta}}{1-\gamma-\zeta}\right]_{q_0}^{q^*} = \frac{a\left(q^*\right) \zeta q_0^{\zeta} \left(q^*\right)^{\gamma}}{1-\gamma-\zeta} \left[\left(q^*\right)^{1-\gamma-\zeta} - q_0^{1-\gamma-\zeta}\right]$$

which is the result. □

B.4 Proof of Proposition 1

Proposition 1. *Consider the task-based production model where the normalized task complexity* $x = q/q_0 \in [1, \infty)$ *is drawn from a continuous distribution with survival function* $\bar{P}(x)$*. Suppose* $\bar{P}(x)$ *is regularly varying at infinity with index* $-\lambda \leq 0$ *(i.e.,* $\bar{P}\left(x\right) = x^{-\lambda} L\left(x\right)$ *for a slowly varying function* L*), or is rapidly varying (*$\lambda = \infty$*). Suppose the aggregate factor ratio* $K/L > 0$ *is fixed, implying the optimal automation cutoff* q^* *is bounded strictly away from zero. As the technological frontier advances (*$q_0 \to 0$*), the aggregate labor share* s_L *satisfies:*

$$\lim_{q_0 \to 0} s_L = \begin{cases} 0 & \text{if } \lambda \geq 1-\gamma, \\ \frac{1-\gamma-\lambda}{1-\gamma} & \text{if } 0 < \lambda < 1-\gamma, \\ 1 & \text{if } \lambda = 0. \end{cases}$$

Proof. From the firm's cost minimization (following the logic in Lemma 2 and Lemma 3), the aggregate cost of capital relative to labor is

$$R\left(q_0, q^*\right) := \frac{rK}{wL} = \frac{\int_{q_0}^{q^*} rq \cdot a\left(q^*\right) \left(\frac{q}{q^*}\right)^{-\gamma} f_{q_0}\left(q\right) dq}{w \cdot a\left(q^*\right) S_{q_0}\left(q^*\right)},$$

where $f_{q_0}\left(q\right) = \frac{1}{q_0} p\left(q/q_0\right)$ is the density and $S_{q_0}\left(q^*\right) = \bar{P}\left(q^*/q_0\right)$ is the survival function. Using the firm's optimal cutoff condition $q^* = w/r$, we can substitute $r/w = 1/q^*$. Making the change of variables $u = q/q_0$, and defining the normalized cutoff $x^* = q^*/q_0$, the ratio simplifies to

$$R(x^*) = \frac{\int_1^{x^*} u^{1-\gamma} p(u)\ du}{(x^*)^{1-\gamma} \bar{P}(x^*)} \equiv \frac{N(x^*)}{D(x^*)},$$

where we define the numerator $N(x^*)$ and the denominator $D(x^*)$ of the fraction. Because aggregate endowments (K, L) are strictly positive, the cutoff q^* is bounded strictly away from zero. Therefore, as the frontier advances ($q_0 \to 0$), the normalized cutoff $x^* \to \infty$. We must therefore evaluate $\lim_{x\to\infty} R(x)$.

We evaluate the numerator $N(x)$ using integration by parts ($p(u) = -\bar{P}'(u)$):

$$N(x) = \bar{P}(1) - x^{1-\gamma}\bar{P}(x) + (1-\gamma)\int_1^x u^{-\gamma}\bar{P}(u)\, du.$$

By hypothesis, $\bar{P}(x)$ is regularly varying with index $-\lambda$. Therefore, the integrand $u^{-\gamma}\bar{P}(u)$ is regularly varying with index $-\gamma - \lambda$. We divide the analysis into three exhaustive cases.

Case 1: Asymptotically Scale-Free Tail ($0 < \lambda < 1 - \gamma$)

Here, $-\gamma - \lambda > -1$. Karamata's Theorem for integration implies

$$\int_1^x u^{-\gamma}\bar{P}(u)\, du \sim \frac{1}{1-\gamma-\lambda} x^{1-\gamma-\lambda} L(x) = \frac{1}{1-\gamma-\lambda} x^{1-\gamma}\bar{P}(x).$$

Substituting this limit back into the expression for $N(x)$, the constant $\bar{P}(1)$ becomes asymptotically negligible, yielding

$$N(x) \sim -x^{1-\gamma}\bar{P}(x) + \frac{1-\gamma}{1-\gamma-\lambda} x^{1-\gamma}\bar{P}(x) = \left(\frac{\lambda}{1-\gamma-\lambda}\right) x^{1-\gamma}\bar{P}(x).$$

Because the denominator is $D(x) = x^{1-\gamma}\bar{P}(x)$, we immediately obtain

$$\lim_{x\to\infty} R(x) = \lim_{x\to\infty} \frac{N(x)}{D(x)} = \frac{\lambda}{1-\gamma-\lambda}.$$

The labor share is $s_L = 1/(1 + R(x))$, which implies $\lim_{q_0\to 0} s_L = \frac{1-\gamma-\lambda}{1-\gamma}$.

Case 2: Thinner Tail ($\lambda \geq 1 - \gamma$)

If $\lambda > 1-\gamma$, then $-\gamma-\lambda < -1$ (this includes the rapidly varying case where $\lambda = \infty$). The function $u^{-\gamma}\bar{P}(u)$ is therefore integrable on $[1, \infty)$, meaning $\int_1^\infty u^{-\gamma}\bar{P}(u)\, du = C < \infty$. Furthermore, because $1 - \gamma - \lambda < 0$, the boundary term $x^{1-\gamma}\bar{P}(x) \to 0$ as $x \to \infty$. (If $\lambda = \infty$, rapid variation implies $\bar{P}(x)$ decays faster than any polynomial, hence $x^{1-\gamma}\bar{P}(x) \to 0$ as well). Consequently, the numerator converges to a strictly positive, finite constant:

$$\lim_{x\to\infty} N(x) = \bar{P}(1) - 0 + (1-\gamma)\, C > 0.$$

Meanwhile, the denominator $D\left(x\right)=x^{1-\gamma}\bar{P}\left(x\right)$ vanishes to 0. Thus, $\lim_{x\to\infty}R\left(x\right)=+\infty$.

If $\lambda=1-\gamma$, we have an index of exactly -1. Here, $D(x)=L(x)$, and by Karamata's theorem, $N(x)\sim(1-\gamma)\tilde{L}(x)$ where $\tilde{L}(x)=\int_1^x u^{-1}L(u)du$ is a new slowly varying function satisfying $L(x)=o(\tilde{L}(x))$. Thus, $R(x)\sim(1-\gamma)\tilde{L}(x)/L(x)\to\infty$.

In both boundary and strict thin-tail cases, the relative cost of capital diverges to infinity, and the labor share collapses: $\lim_{q_0\to 0}s_L=0$.

Case 3: Thicker Tail ($\lambda=0$)

Here, $\bar{P}\left(x\right)=L\left(x\right)$ is a slowly varying function. The denominator $D\left(x\right)=x^{1-\gamma}L\left(x\right)\to\infty$ because $1-\gamma>0$. For the numerator, we apply Karamata's Theorem for index $-\gamma>-1$:

$$\int_1^x u^{-\gamma}\bar{P}\left(u\right)du=\frac{1}{1-\gamma}x^{1-\gamma}\bar{P}\left(x\right)+o\left(x^{1-\gamma}\bar{P}\left(x\right)\right).$$

Substituting this into $N\left(x\right)$ makes the leading-order terms cancel exactly:

$$\begin{aligned}N\left(x\right)&=\bar{P}\left(1\right)-x^{1-\gamma}\bar{P}\left(x\right)+\left(1-\gamma\right)\left[\frac{1}{1-\gamma}x^{1-\gamma}\bar{P}\left(x\right)+o\left(x^{1-\gamma}\bar{P}\left(x\right)\right)\right]\\&=\bar{P}\left(1\right)\underbrace{-x^{1-\gamma}\bar{P}\left(x\right)+x^{1-\gamma}\bar{P}\left(x\right)}_{=0}+o\left(x^{1-\gamma}\bar{P}\left(x\right)\right).\end{aligned}$$

Because $D(x)=x^{1-\gamma}\bar{P}(x)\to\infty$, the constant $\bar{P}(1)$ is also absorbed into the little-o term, leaving $N(x)=o\big(x^{1-\gamma}\bar{P}\left(x\right)\big)$. Because $N\left(x\right)$ is strictly $o\left(D\left(x\right)\right)$ and $D\left(x\right)\to\infty$, it follows that $\lim_{x\to\infty}R\left(x\right)=0$. The labor share thus converges to unity: $\lim_{q_0\to 0}s_L=1$. □

B.5 Proof of Proposition 2

Proposition 2. *Let $b>0$ and define $\mathcal{D}_b=\{(K,L):K/L>b\}$. As $q_0\to 0$, the task-based production function $F\left(K,L;q_0\right)$ converges on $\mathcal{D}_b$ to the Cobb-Douglas limit uniformly in relative error. That is*

$$\lim_{q_0\to 0}\sup_{(K,L)\in\mathcal{D}_b}\left|\frac{F(K,L;q_0)}{F^{CD}(K,L;q_0)}-1\right|=0,$$

where the constant returns to scale Cobb-Douglas production function is given by

$$F^{CD}\left(K,L;q_0\right):=\mathcal{A}\left(q_0\right)^{-(1-\alpha)}L^{\alpha}K^{1-\alpha},\tag{13}$$

and where $\mathcal{A}>0$ is a constant that only depends on parameters. Furthermore, the first derivatives of F converge in the analogous way to their F^{CD} counterparts.

Proof. Consider a firm that is endowed with $K>0$ units of capital and $L>0$ units of labor, such that $K/L>b$. This firm allocates tasks to the factors to maximize output. This implies setting $q^*=w/r$ where w and r are the shadow costs of the factors (Lagrange multipliers). The equations of Lemma 3 hold, and so

$$Y = \frac{L}{a\left(q^*\right)\left(\frac{q_0}{q^*}\right)^{\zeta}} \text{ and } Y = \frac{K}{q^* a\left(q^*\right)\zeta\left(\frac{q_0}{q^*}\right)^{\zeta}\frac{1}{1-\gamma-\zeta}\left(1-\left(\frac{q_0}{q^*}\right)^{1-\gamma-\zeta}\right)}.$$

We can therefore write the output of the firm as

$$Y = Y^{\alpha}Y^{1-\alpha} = \frac{1}{a\left(q^*\right)\left(\frac{q_0}{q^*}\right)^{\zeta}}\left(\frac{1}{q^*\frac{\zeta}{1-\gamma-\zeta}\left(1-\left(\frac{q_0}{q^*}\right)^{1-\gamma-\zeta}\right)}\right)^{1-\alpha} L^{\alpha}K^{1-\alpha}, \tag{35}$$

where we recognize the shape of the Cobb-Douglas production function with a total factor productivity term that depends on the (endogenous) threshold q^* and the (exogenous) technological frontier q_0. Simple algebra implies that the ratio of the production function is equal to

$$\frac{F\left(K,L;q_0\right)}{F^{CD}\left(K,L;q_0\right)} = \frac{1}{\mathcal{A}}\frac{1}{A\left(\frac{1-\gamma}{1-\gamma-\zeta}-\frac{\zeta}{1-\gamma-\zeta}\left(\frac{q_0}{q^*}\right)^{1-\gamma-\zeta}\right)^{\frac{\gamma}{1-\gamma}}}\left(\frac{1}{\frac{\zeta}{1-\gamma-\zeta}\left(1-\left(\frac{q_0}{q^*}\right)^{1-\gamma-\zeta}\right)}\right)^{1-\alpha} \tag{36}$$

where we have used the definition of $a\left(q^*\right)$.

Next, notice that we can write (12) as

$$\frac{K}{L} = \frac{\zeta}{1-\gamma-\zeta}q^*\left[1-\left(\frac{q_0}{q^*}\right)^{1-\gamma-\zeta}\right]. \tag{37}$$

Since $(K,L)\in\mathcal{D}_b$, it must be that q^* is bounded away from zero for all q_0.

To prove uniform convergence, we must show that for all $\varepsilon>0$ there exists a $\bar{q}_0$ such that for all $q_0<\bar{q}_0$ the inequality

$$\left|\frac{F\left(K,L;q_0\right)}{F^{CD}\left(K,L;q_0\right)}-1\right|<\varepsilon$$

holds for all $(K,L)\in\mathcal{D}_b$. Since q^* is bounded away from 0 for all $(K,L)\in\mathcal{D}_b$, this follows directly from (36). Indeed, that expression shows that F/F^{CD} does not depend on (K,L) and that it is strictly increasing in q_0.

Since $K/L>b>0$, q^* is bounded strictly away from zero for any q_0 (see (37)). Consequently, q^* cannot converge to zero as $q_0\to 0$. This implies that we can find $\mathcal{A}>0$ such that for any $\varepsilon>0$, there exist a $q_0>0$ small enough such that

$$\left|\frac{F\left(K,L;q_0\right)}{F^{CD}\left(K,L;q_0\right)}-1\right|<\varepsilon$$

for every $(K, L) \in \mathcal{D}_b$. As a result, F converges uniformly to F^{CD} on $\mathcal{D}_b$.

The first derivatives of the task-based production function also converge uniformly to their Cobb-Douglas counterpart. First, notice that F is differentiable on $\mathcal{D}_b$. Indeed, F is given by (35) where q^* follows (37). A marginal change in K or L therefore implies a smooth change in q^* and in F. Next, we can write

$$\frac{\partial F}{\partial L} = s_L\left(\frac{q_0}{q^*}\right)\frac{F\left(K, L; q_0\right)}{L} \text{ and } \frac{\partial F^{CD}}{\partial L} = \alpha\frac{F^{CD}\left(K, L; q_0\right)}{L},$$

and the result follows from the uniform convergence of F to F^{CD} and of s_L to α. A similar argument applies for the marginal product of capital. □

B.6 Proof of Lemma 4

Lemma 4. *Suppose that $q^* > q_0$. The elasticity of substitution between capital and labor for the task-based technology is given by*

$$\sigma_{KL} = 1 + \frac{\beta\left(\frac{q_0}{q^*}\right)^{\beta}}{1-\left(\frac{q_0}{q^*}\right)^{\beta}} \geq 1,$$

where $\beta = 1-\gamma-\zeta > 0$. Consequently, the elasticity of substitution satisfies $\sigma_{KL} \geq 1$ and converges to the Cobb-Douglas unitary value $\sigma_{KL} \to 1$ as $q_0 \to 0$.

Proof. The firm's cost minimization implies that the optimal automation cutoff satisfies $q^* = w/r$. Let $k := K/L$. From Lemma 3, the relationship between capital deepening and the cutoff is given by

$$k = \frac{\zeta}{\beta}q^*\left[1-\left(\frac{q_0}{q^*}\right)^{\beta}\right].$$

Let $x := q_0/q^*$ denote the proximity to the frontier. Taking the logarithm of the equation yields

$$\ln k = \ln\left(\frac{\zeta}{\beta}\right) + \ln q^* + \ln(1-x^{\beta}).$$

We differentiate with respect to $\ln(w/r) = \ln q^*$. Note that $x = q_0(q^*)^{-1}$, so $\frac{d\ln x}{d\ln q^*} = -1$.

$$\frac{d\ln k}{d\ln q^*} = 1 + \frac{d\ln(1-x^{\beta})}{d\ln q^*} = 1 + \frac{1}{1-x^{\beta}}\frac{d(1-x^{\beta})}{dx}\frac{dx}{d\ln q^*}.$$

The derivative term is:

$$\frac{d(1-x^{\beta})}{dx} = -\beta x^{\beta-1}.$$

Using $\frac{dx}{d\ln q^*} = -x$:

$$\frac{d\ln k}{d\ln q^*} = 1 + \frac{1}{1-x^\beta}(-\beta x^{\beta-1})(-x) = 1 + \frac{\beta x^\beta}{1-x^\beta}.$$

Since $\beta > 0$ and $x \in (0,1)$, the second term is strictly positive. Thus $\sigma > 1$. As the economy develops ($q_0/q^* \to 0$), $x \to 0$, and the second term vanishes, yielding $\sigma_{KL} = 1$. □

B.7 Proof of Proposition 3

Proposition 3. *Consider the task-based production model with a capital requirement distribution characterized by a continuous probability density function $\tilde{f}(q)$ with support $[q_0, \infty)$, where $q_0 > 0$. There exists no distribution $\tilde{f}$ such that the associated production function $F(K, L)$ is of the Cobb-Douglas form for all factor endowments $(K, L) \in \mathbb{R}^2_{++}$.*

Proof. We proceed by contradiction. We first assume there exists a distribution with density $\tilde{f}(q)$ supported on $[q_0, \infty)$ such that the aggregate production function is Cobb-Douglas.

Step 1: Constant factor share condition.

A necessary and sufficient condition for a production function to be globally Cobb-Douglas is that the ratio of factor income shares is constant for all K, L. Let $\theta > 0$ be a positive constant. We require

$$\frac{rK}{wL} = \theta \quad \forall (K, L).$$

In the task-based model, the optimal automation cutoff is determined by $q^* = w/r$. Thus, the condition implies that for all $q^* > q_0$,

$$\frac{K(q^*)}{L(q^*)} = \theta q^*.$$

Using the optimal task intensity profile derived in Lemma 2, where $a(q) = a(q^*)(q/q^*)^{-\gamma}$ for automated tasks and $a(q) = a(q^*)$ for labor tasks, we can express the aggregate factor demands as[13]

$$L(q^*) = a(q^*) \int_{q^*}^{\infty} \tilde{f}(q)\, dq = a(q^*)[1 - \tilde{F}(q^*)],$$

$$K(q^*) = \int_{q_0}^{q^*} q \cdot a(q) \tilde{f}(q)\, dq = a(q^*)(q^*)^\gamma \int_{q_0}^{q^*} q^{1-\gamma} \tilde{f}(q)\, dq.$$

Substituting these into the ratio condition, the baseline intensity $a(q^*)$ cancels out

$$\frac{a(q^*)(q^*)^\gamma \int_{q_0}^{q^*} q^{1-\gamma} \tilde{f}(q)\, dq}{a(q^*)\left[1 - \tilde{F}(q^*)\right]} = \theta q^*.$$

[13] Note that this equation for $a(q)$ does not rely on the underlying task distribution being Pareto.

Let $S(q) := 1 - \tilde{F}(q)$ denote the survival function. The condition simplifies to:

$$\int_{q_0}^{q^*} q^{1-\gamma} \tilde{f}(q)\, dq = \theta (q^*)^{1-\gamma} S(q^*) . \tag{38}$$

Step 2: Necessary condition on distribution.

Differentiate (38) with respect to q^* to solve for the density f. The LHS derivative is $(q^*)^{1-\gamma}\tilde{f}(q^*)$ by the Fundamental Theorem of Calculus. The RHS derivative is $\theta\left[(1-\gamma)(q^*)^{-\gamma}S(q^*) + (q^*)^{1-\gamma}S'(q^*)\right]$. Recalling that $S'(q^*) = -\tilde{f}(q^*)$, we have:

$$(q^*)^{1-\gamma}\tilde{f}(q^*) = \theta(1-\gamma)(q^*)^{-\gamma}S(q^*) - \theta(q^*)^{1-\gamma}\tilde{f}(q^*).$$

Rearranging terms to group $\tilde{f}(q^*)$:

$$(1+\theta)(q^*)^{1-\gamma}\tilde{f}(q^*) = \theta(1-\gamma)(q^*)^{-\gamma}S(q^*).$$

Dividing by $(q^*)^{1-\gamma}$ (since $q^* \geq q_0 > 0$) and substituting $f(q^*) = -S'(q^*)$:

$$-(1+\theta)S'(q^*) = \theta(1-\gamma)\frac{1}{q^*}S(q^*).$$

This is a separable first-order differential equation:

$$\frac{S'(q^*)}{S(q^*)} = -\frac{\theta(1-\gamma)}{1+\theta}\frac{1}{q^*}.$$

Integrating yields $S(q^*) = C(q^*)^{-\lambda}$, where $\lambda = \frac{\theta(1-\gamma)}{1+\theta} > 0$. This implies that the distribution *must* be Pareto (or power-law) with tail index λ.

Step 3: Contradiction.

We have established that a Pareto distribution is the only candidate. Let $\tilde{f}(q) = \lambda q_0^\lambda q^{-\lambda-1}$ for $q \geq q_0$. We substitute this density back into the LHS of the necessary condition (38):

$$\text{LHS} = \int_{q_0}^{q^*} q^{1-\gamma}(\lambda q_0^\lambda q^{-\lambda-1})\, dq = \lambda q_0^\lambda \int_{q_0}^{q^*} q^{1-\gamma-\lambda-1}\, dq.$$

Since $\lambda \neq 1-\gamma$, the integral evaluates to

$$\text{LHS} = \lambda q_0^\lambda \left[\frac{q^{1-\gamma-\lambda}}{1-\gamma-\lambda}\right]_{q_0}^{q^*} = \frac{\lambda q_0^\lambda}{1-\gamma-\lambda}\left[(q^*)^{1-\gamma-\lambda} - q_0^{1-\gamma-\lambda}\right].$$

Now we construct the ratio $\frac{\text{LHS}}{\text{RHS}}$ utilizing the RHS $\theta(q^*)^{1-\gamma}S(q^*) = \theta(q^*)^{1-\gamma}(q_0/q^*)^{\lambda}$:

$$\frac{\text{LHS}}{\text{RHS}} \propto \frac{(q^*)^{1-\gamma-\lambda} - q_0^{1-\gamma-\lambda}}{(q^*)^{1-\gamma-\lambda}} = 1 - \left(\frac{q_0}{q^*}\right)^{1-\gamma-\lambda}.$$

For the production function to be Cobb-Douglas, this ratio must be identically equal to 1 for all $q^* > q_0$. However, the term $\left(\frac{q_0}{q^*}\right)^{1-\gamma-\lambda}$ varies with q^*. Specifically, since $q_0 > 0$, this term is non-zero and depends on the cutoff (and thus on factor endowments). The ratio equals 1 only if $q_0 = 0$. Since the premise requires a positive frontier $q_0 > 0$, condition (38) cannot be satisfied for all q^*. Thus, exact Cobb-Douglas aggregation is impossible for any positive $q_0 > 0$. □

B.8 Proof of Proposition 4

Proposition 4. *Suppose $K_0 > 0$. The capital per effective worker in the task-based economy $\hat{k}_t$ converges to the unique steady state of the Cobb-Douglas economy along the optimal equilibrium path:*

$$\lim_{t\to\infty} \hat{k}_t = \hat{k}^*_{CD}, \tag{14}$$

Furthermore, the asymptotic growth rates of K_t, C_t and Y_t converge to ν.

Proof. The proof proceeds in four steps. First we define the dynamic system followed by the task-based economy. Second, we do the same for Cobb-Douglas. Third, we show that the task-based system converges uniformly to the Cobb-Douglas system. Fourth, we rely on the theory of asymptotically autonomous differential equations to show the convergence of the stationary point.

Step 1. The detrended dynamic system.

We first construct the dynamic system followed by the task-based economy. Differentiating $\hat{k}_t$ with respect to time and combining with the resource constraint (1), we get the law of motion for capital

$$\dot{\hat{k}}_t = \frac{F\left(K_t, L; q_0\left(t\right)\right)}{Le^{\nu t}} - \hat{c}_t - \left(\delta + \nu\right)\hat{k}_t. \tag{39}$$

Similarly, from the household's Euler equation

$$\frac{\dot{C}}{C} = F_K\left(K_t, L_t; q_0\left(t\right)\right) - \rho - \delta,$$

we obtain the equation of motion for consumption

$$\frac{\dot{\hat{c}}_t}{\hat{c}_t} = F_K\left(K_t, L_t; q_0\left(t\right)\right) - \left(\delta + \rho + \nu\right). \tag{40}$$

The pair (39) and (40) defines a *non-autonomous planar dynamical system*, which we denote by $\dot{x} = G\left(x, t\right)$ with $x = \left(\hat{k}, \hat{c}\right)$. Note that it is clear from (40) already that $\hat{k}^*_{CD}$ is an asymptotic steady state of G, since it makes the right-hand side of that equation equal to zero as $t \to \infty$.

Step 2. Limiting autonomous system.

Next, we define the limiting *autonomous* system $\dot{x} = G_\infty(x)$, corresponding to the Cobb-Douglas production function (13). Substituting this functional form into the detrended equations yields

$$\dot{\hat{k}} = \mathcal{A}\hat{k}^{1-\alpha} - \hat{c} - (\delta+\nu)\hat{k}, \quad \text{and} \quad \frac{\dot{\hat{c}}}{\hat{c}} = (1-\alpha)\mathcal{A}\hat{k}^{-\alpha} - (\delta+\rho+\nu).$$

This system is time-independent. Standard growth theory establishes that it admits a unique steady state $x^* = \left(\hat{k}^*_{CD}, \hat{c}^*_{CD}\right)$ with $\hat{k}^*_{CD} > 0$. As is well-known, the Jacobian matrix associated with G_∞ evaluated at x^* possesses one positive and one negative real eigenvalue (saddle-point stability).

Step 3. Uniform convergence of the vector field.

We now show that $G(x,t) \to G_\infty(x)$ uniformly on compact sets $\mathcal{D} = \left[\underline{k}, \bar{k}\right] \times [\underline{c}, \bar{c}] \subset \mathbb{R}^2_{++}$. Define $K_t\left(\hat{k}\right) := Le^{\nu t}\hat{k}$. For $\hat{k} \in [\underline{k}, \bar{k}]$, we have $K_t\left(\hat{k}\right) \in \left[Le^{\nu t}\underline{k}, Le^{\nu t}\bar{k}\right]$. For $b < \bar{\kappa}$, this interval lies within the domain of uniform relative convergence in Proposition 2.

Fix $\left(\hat{k}, \hat{c}\right) \in \mathcal{D}$ and define the relative error term

$$\varepsilon_t\left(\hat{k}\right) := \frac{F\left(K_t\left(\hat{k}\right), L; q_0(t)\right) - F^{CD}\left(K_t\left(\hat{k}\right), L; q_0(t)\right)}{F^{CD}\left(K_t\left(\hat{k}\right), L; q_0(t)\right)}.$$

We know from Proposition 2 that

$$\sup_{\hat{k} \in \left[\underline{k}, \bar{k}\right]} \left|\varepsilon_t\left(\hat{k}\right)\right| \to 0.$$

Now rewrite,

$$\frac{F\left(K_t\left(\hat{k}\right), L; q_0(t)\right)}{Le^{\nu t}} = \frac{F^{CD}\left(K_t\left(\hat{k}\right), L; q_0(t)\right)}{Le^{\nu t}}\left(1 + \varepsilon_t\left(\hat{k}\right)\right).$$

We have already shown above that $\frac{F^{CD}\left(K_t\left(\hat{k}\right), L; q_0(t)\right)}{Le^{\nu t}} = \mathcal{A}\hat{k}^{1-\alpha}$ and so since $\hat{k} \in \left[\underline{k}, \bar{k}\right]$ it follows that

$$\mathcal{A}\underline{k}^{1-\alpha} \leq \frac{F\left(K_t, L; q_0(t)\right)}{Le^{\nu t}} \leq \mathcal{A}\bar{k}^{1-\alpha}.$$

Next, consider the first element in G (the law of motion of capital). From the definition of G and G_∞ we can write

$$\left|G_1(x,t) - G_{1,\infty}(x)\right| = \left|\frac{F^{CD}\left(K_t, L; q_0(t)\right)}{Le^{\nu t}}\varepsilon_t\left(\hat{k}\right)\right|.$$

We can bound this term as

$$\sup_{x \in \mathcal{D}} \left|G_1(x,t) - G_{1,\infty}(x)\right| \leq \mathcal{A}\bar{k}^{1-\alpha} \sup_{\hat{k} \in \left[\underline{k}, \bar{k}\right]} \left|\varepsilon_t\left(\hat{k}\right)\right| \xrightarrow[t\to\infty]{} 0,$$

and so the first element in G converges uniformly in levels to $G_{1,\infty}$. An analogous argument applies to the Euler equation term in G_2 (using the uniform convergence of the marginal product). Thus, the full function G converges uniformly to G_∞:

$$\lim_{t\to\infty}\sup_{x\in\mathcal{D}}\left|G\left(x,t\right)-G_\infty\left(x\right)\right|=0.$$

Step 4. Convergence of the optimal trajectory.

We now apply the theory of asymptotically autonomous systems to establish convergence. We have shown uniform convergence $G(x,t)\to G_\infty(x)$ on compact sets. The limiting system $\dot{x}=G_\infty(x)$ has a unique steady state $x^*=(\hat{k}^*_{CD},\hat{c}^*_{CD})$, which is a hyperbolic saddle point. By Theorem 4.1 in Thieme (1992), the ω-limit of any bounded solution $x(t)$ of the non-autonomous system $\dot{x}=G(x,t)$ is contained in the equilibrium set of the limit autonomous system.[14] Furthermore, the transversality condition

$$\lim_{t\to\infty}e^{-\rho t}u'(C_t)K_t=0$$

eliminates any trajectories that diverge along the unstable manifold of the saddle point. Since the stable manifold is unique and leads to x^*, the unique optimal equilibrium trajectory must satisfy $\lim_{t\to\infty}x\left(t\right)=x^*$. □

B.9 Proof of Proposition 5

Proposition 5. *For $t>T_{takeoff}$, the competitive equilibrium of the task-based economy is governed by*

$$\dot{K}_t=y\left(u_t\right)L-C_t-\delta K_t, \tag{17}$$

$$\frac{\dot{C}_t}{C_t}=r\left(u_t,q_0\left(t\right)\right)-\rho-\delta, \tag{18}$$

where $q_0\left(t\right)=e^{-\phi g t}$ and $u_t>1$ is the unique solution to

$$\frac{K_t}{L}=\frac{\zeta\,q_0\left(t\right)}{\beta}\,u_t\left(1-u_t^{-\beta}\right). \tag{19}$$

Output per worker and the rental rate of capital are given by

$$y\left(u\right)=\frac{1}{A}u^{\frac{\zeta}{1-\gamma}}\left[\frac{\beta}{(1-\gamma)-\zeta\,u^{-\beta}}\right]^{\frac{\gamma}{1-\gamma}}, \tag{20}$$

$$r\left(u,q_0\right)=\frac{1}{Aq_0}u^{-\alpha}\left[\frac{\beta}{(1-\gamma)-\zeta\,u^{-\beta}}\right]^{\frac{1}{1-\gamma}}. \tag{21}$$

[14] Because of the decreasing returns to capital, the feasibility constraint and the resource constraint, any solution $x\left(t\right)$ must be bounded.

Along the transition, the labor share is given by

$$s_L(u) = \frac{\beta}{(1-\gamma) - \zeta\, u^{-\beta}}. \tag{22}$$

Proof. The proof proceeds in three steps. First, we express output per worker as a function of the automation depth u. Second, we derive the marginal product of capital. Third, we establish the implicit equation for u and verify uniqueness.

Step 1. Output per worker.

From Lemma 3, labor demand per unit of output is $L/Y = a(q^*)(q_0/q^*)^{\zeta}$. Inverting:

$$\frac{Y}{L} = \frac{1}{a(q^*)(q_0/q^*)^{\zeta}}.$$

We re-express each term using $u = q^*/q_0$, so that $(q_0/q^*)^{\zeta} = u^{-\zeta}$ and $(q_0/q^*)^{1-\gamma} = u^{-(1-\gamma)}$. From Lemma 2, the intensity at the cutoff is

$$a(q^*) = A\left[\frac{(1-\gamma)\,u^{-\zeta} - \zeta u^{-(1-\gamma)}}{\beta}\right]^{\frac{\gamma}{1-\gamma}}.$$

Since $(1-\gamma) - \zeta = \beta$, we can write $u^{-(1-\gamma)} = u^{-\zeta} \cdot u^{-\beta}$, giving

$$a(q^*) = A\left[\frac{u^{-\zeta}\left((1-\gamma) - \zeta u^{-\beta}\right)}{\beta}\right]^{\frac{\gamma}{1-\gamma}}.$$

Therefore,

$$a(q^*) \cdot u^{-\zeta} = A \cdot u^{-\zeta\left(1+\frac{\gamma}{1-\gamma}\right)}\left[\frac{(1-\gamma) - \zeta u^{-\beta}}{\beta}\right]^{\frac{\gamma}{1-\gamma}} = A \cdot u^{-\frac{\zeta}{1-\gamma}}\left[\frac{(1-\gamma) - \zeta u^{-\beta}}{\beta}\right]^{\frac{\gamma}{1-\gamma}},$$

where we used $1 + \gamma/(1-\gamma) = 1/(1-\gamma)$. Inverting yields (20).

Step 2. Marginal product of capital.

In competitive equilibrium, $F_K = r$ where r is the rental rate. Since $q^* = w/r$, we have $r = w/q^* = w/(q_0 u)$. The wage satisfies $w = s_L \cdot Y/L = s_L \cdot y(u)$. Hence,

$$F_K = \frac{s_L(u) \cdot y(u)}{q_0\, u}.$$

From (10), the labor share expressed in terms of u is

$$s_L(u) = \left[1 + \frac{\zeta}{\beta}\left(1 - u^{-\beta}\right)\right]^{-1} = \frac{\beta}{(1-\gamma) - \zeta u^{-\beta}},$$

where we combined the terms as $1 + \zeta/\beta - \zeta u^{-\beta}/\beta = \left[(1-\gamma) - \zeta u^{-\beta}\right]/\beta$. Multiplying $s_L(u) \cdot y(u)/(q_0 u)$: the two bracket terms combine with exponents $1 + \gamma/(1-\gamma) = 1/(1-\gamma)$, and the power of u is

$$\frac{\zeta}{1-\gamma} - 1 = \frac{\zeta - 1 + \gamma}{1-\gamma} = -\frac{\beta}{1-\gamma} = -\alpha,$$

yielding (21).

Step 3. Implicit equation and uniqueness.

The implicit equation (19) is simply the capital-labor ratio (12) rewritten in terms of u:

$$\frac{K}{L} = \frac{\zeta}{\beta} q^* \left[1 - \left(\frac{q_0}{q^*}\right)^{\beta}\right] = \frac{\zeta q_0}{\beta} u \left(1 - u^{-\beta}\right).$$

Define $h(u) := u\left(1 - u^{-\beta}\right) = u - u^{1-\beta}$ for $u \geq 1$. Its derivative is

$$h'(u) = 1 - (1-\beta) u^{-\beta} = 1 - (\gamma + \zeta) u^{-\beta}.$$

Since $\gamma + \zeta < 1$ and $u \geq 1$, we have $h'(u) \geq \beta > 0$, so h is strictly increasing. Moreover, $h(1) = 0$ and $h(u) \to \infty$ as $u \to \infty$. It follows that for any $K/L > 0$ and $q_0 > 0$, the equation $\frac{\zeta q_0}{\beta} h(u) = K/L$ admits a unique solution $u > 1$, which can be computed by standard root-finding algorithms. □