

Econ 871: LECTURE NOTES

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Contents

| | | |
|----------|--|-----------|
| 1 | International Trade | 1 |
| 1.1 | Introduction | 1 |
| 1.2 | Patterns of Trade in the Aggregate Data | 2 |
| 1.3 | Armington Model | 3 |
| 1.4 | Dornbusch, Fisher and Samuelson Model | 22 |
| 1.5 | Heckscher-Ohlin Model | 24 |
| 1.6 | Eaton and Kortum Model | 26 |
| 1.7 | New Trade Theory | 57 |
| 1.8 | Krugman-Helpman Model | 57 |
| 1.9 | Melitz Model | 64 |
| 1.10 | Bernard, Eaton, Jensen and Kortum Model | 76 |
| 2 | International Business Cycle | 83 |
| 2.1 | Introduction | 83 |
| 2.2 | Business Cycle Measurement | 86 |
| 2.3 | Prototype International Business Cycle Model | 87 |
| 2.4 | Basic Supply-Side Extensions | 114 |
| 2.5 | Calibration | 121 |
| 2.6 | Quantitative Comparison of Theory and Data | 154 |
| 2.7 | Drozdz and Nosal (2008) | 162 |

| | |
|---|------------|
| 2.8 Other Extensions | 162 |
| 3 Papers for Student Presentations | 165 |

List of Figures

| | | |
|-----|---|-----|
| 1.1 | An example of isolated small country, and implications for aggregate price level. | 12 |
| 1.2 | Canadian Producers and Consumers Pay the Iceberg Cost. | 14 |
| 1.3 | Frechet density function. | 28 |
| 1.4 | Price schedules and specialization pattern in the EK model. | 43 |
| 1.5 | Price schedules and specialization in the Armington model. | 45 |
| 1.6 | Decomposition of the distribution. | 52 |
| 1.7 | Industry level relocations in the Melitz model. | 75 |
| 2.1 | Time-horizon of the business cycle analysis. | 84 |
| 2.2 | Comparison of real exchange rate with terms of trade (linearly detrended data). | 113 |
| 2.3 | Share of working age population and population growth in the US. . . | 124 |
| 2.4 | Internal structure of NIPAs. | 130 |
| 2.5 | GDI (=GDP + statistica discrepancy) broken down by income type. . | 131 |
| 2.6 | Calibrated values of α , 1959-2005. | 132 |
| 2.7 | Calibrated values of δ , 1959-2006. | 134 |
| 2.8 | Calibrated values of η , 1959-2006. | 136 |
| 2.9 | Calibrated values of β for $\theta = 2$, 1959-2006. | 137 |

| | |
|---|-----|
| 2.10 Comparison of K/Y ratio between model (BGP path) and US data (initial value matched by construction). | 145 |
| 2.11 Basic Properties of International Business Cycles: Volatility and Co- movement Patterns. | 156 |

List of Tables

| | | |
|-----|---|-----|
| 1.1 | Comparing CA-US and US-CA gravity equations. | 17 |
| 2.1 | Real export and real import prices.s ^a | 114 |
| 2.2 | Relative volatility of the terms of trade. | 115 |
| 2.3 | Summary of the puzzles on prices. | 116 |
| 2.4 | Parameter values for the quarterly model. | 138 |
| 2.5 | Volatility Ratio in a Cross-Section of Countries. | 141 |
| 2.6 | Comparison of Models with Data. | 164 |

Chapter 1

International Trade

1.1 Introduction

International trade theory addresses the question why in the long-run countries engage in the exchange of goods and services. Depending on the answer, models of trade can be classified into three major categories: (i) Ricardian models, (ii) Heckscher-Ohlin models, and (iii) Monopolistic competition models.

The first two categories are referred to as the *traditional trade theory*. In these models, countries trade because there are intrinsically different, and by the logic of Ricardian comparative advantage, trade allows them to take advantage of these differences. In particular, in the Ricardian models the technologies to produce each good differ, and in the Heckscher-Ohlin models, the factor endowments differ. Both features result in a situation of comparative advantage, and lead to a partial or a complete specialization.

The monopolistic competition models, referred to as the *new trade theory*, depart from this traditional approach in at least two important respects. First, in the new trade theory countries are no longer intrinsically different (ex-ante), but still trade and specialize (ex-post). The key idea is that trade and specialization allows them

to access a larger variety of goods, while at the same time exploit economies of scale present in producing them.

The second important difference is the modeling methodology. In contrast to the traditional trade theory, the new trade theory is a strictly positive theory¹. Namely, it attempts to directly describe and mirror the exact market and institutional structure that we see in the data, and is silent about the deep-rooted frictions that could give rise to this structure endogenously. The weakness of such approach is the need for a more extensive data-based justification of its more complex structure, but its strength is a more natural mapping between the theory and the data. In trade theory, it has paid off by allowing researchers to extend trade facts to producer-level facts, and as Samuel Kortum puts it, *“by building on the firm-level stylized facts, the resulting aggregate theory is likely to be more credible both as a description of reality and as a tool for policy analysis.”*

We should also mention that initially the new trade theory was a theoretical response to the empirical observation that most trade takes place between very similar countries (industrial countries), and most importantly, the observation that these countries tend to trade very similar categories of manufactured goods (called intra-industry trade). Popular at the time Heckscher-Ohlin models could not sensibly deal with observation. In this respect, monopolistic competition models still have an edge over other theories. Even though Ricardian models can sensibly deal with intra-industry trade, they are silent about the source of the underlying technology differences that lead to this phenomenon. In the future, the ongoing integration of the Ricardian theory with the theory of innovation and growth is likely to fill this gap.

1.2 Patterns of Trade in the Aggregate Data

[to be completed]

¹Positive theory directly characterizes what is. Normative theory focuses on what ought to be.

1.3 Armington Model

Armington model is a Ricardian model.² Each country in the Armington model is assumed to be efficient in producing just one good, and infinitely inefficient in producing all the other. This assumption makes the comparative advantage structure somewhat trivial, but the model becomes very tractable.

In this section, we will study the predictions of a basic multicountry Armington model, and apply them to understand trade flows between countries.

Model Economy

There are N countries (or regions) and N goods in the world. Each country has the technology to produce only one good from the set $1, \dots, N$, and can not produce all the other goods by assumption. Production factors are assumed immobile across countries, and all markets are competitive. In terms of notation, we assume that country n produces good n .

Geography is modeled here by an iceberg transportation cost that is intended to capture the notion of trade barriers between countries (regions). Iceberg transportation cost d_{ni} between country n and i implies that d_{ni} units of good must be shipped from country n in order for one unit to arrive in country i . In what follows, the following properties of the iceberg cost will be assumed: (i) symmetry $d_{ni} = d_{in} \geq 1$, (ii) no cost within the country $d_{ii} = 1$, (iii) triangle inequality

$$d_{ni} \leq d_{nj} + d_{ji} \quad \text{all } i, j, n = 1..N.$$

²Since this model is the baseline framework adopted in the open economy macro, it is particularly important to know how it fits into the broadly defined trade theory.

Households

In each country $n = 1, \dots, N$, there is a stand-in household that has preferences described by a CES aggregator given by

$$U_n = \left(\sum_{i=1..N} \alpha_i^{\frac{1}{\sigma}} c_{ni}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \quad (1.1)$$

where σ is the elasticity of substitution between the goods ($\sigma > 1$), and α_i is the weight of each good ($\sum_i \alpha_i = 1$). Each household is assumed to inelastically supply its endowment of L_n units of labor.

Given market wage w_i , and a schedule of prices p_{ni} for each good, the problem of the household (in country n) is to maximize (1.1) subject to the budget constraint given by

$$\sum_{i=1..N} p_{ni} c_{ni} = w_n L_n + \Pi_n, \quad (1.2)$$

where Π_n are the profits paid out by the local firms (in equilibrium Π_n will be zero).

Firms

In each country, there is a stand-in competitive firm that takes all prices as given. The firm employs labor supplied by the home households, produces goods, and sells these goods both at home and abroad. Production technology is assumed to be subject to constant returns to scale.

The profit function of a stand-in firm from country i is given by

$$\Pi_i = \sum_{n=1..N} p_{ni} y_{ni} - w_i l_i, \quad (1.3)$$

where y_{ni} is the amount of good i sold in each country n (sold there at price p_{ni}), and l_i is labor input. The firm's objective is to maximize (1.3) subject to production

constraint

$$\sum_{i=1..N} d_{ni} y_{ni} \leq l_i, \quad (1.4)$$

where the left-hand-side denotes the total quantity produced, and the right hand side is the production function.

Market Clearing and Feasibility

Market clearing requires that the supply of each good equals the demand for each good,

$$c_{ni} = y_{ni}, \text{ all } n, i. \quad (1.5)$$

and the supply of labor equals to the demand for labor,

$$L_n = l_n, \text{ all } n. \quad (1.6)$$

Equilibrium

The definition of equilibrium is as follows.

Definition 1 *Competitive equilibrium in this economy is:*

- prices w_n, p_{ni} ,
- and allocation c_{ni}, y_{ni}, l_n ,

such that

- given prices, c_{ni} solves the household's problem,
- given prices, y_{ni}, l_n , solve the firm's problem,
- and all markets clear.

Proposition 2 *The competitive equilibrium allocation exists and is unique.*

Proof. Competitive equilibrium allocation, if exists, must be Pareto efficient by 1st Welfare Theorem, and so the allocation must solve the planning problem given by maximization of (1.1) subject to (1.4), (1.5) and (1.6). Since this planning problem involves a maximization of a continuous and concave objective function subject to a convex and compact constraint set, the solution to the planning problem is unique. Thus, by 2nd Welfare Theorem, the competitive equilibrium exists and is unique. ■

Exercise 3 *Show that in equilibrium the following version of the law of one price must hold:*

$$p_{ni} = d_{ni}p_{ii}, \text{ all } n, i = 1, \dots, N.$$

Predictions for Trade

In its general formulation, the Armington model can not be solved analytically, and so we have to resort to a partial characterization of the equilibrium. The proposition below derives the model's key predictions the patterns of trade and how they depend on geography. We will refer to this prediction as the *gravity equation*.³ In general, the gravity equation is an equation characterizing how trade shares (expenditure shares of one country on some other country's goods) are related to income levels, and various measures capturing trade costs.

Proposition 4 *In the Armington model, the share of expenditures of country n on goods imported from country i in total expenditures of country n is given by the fol-*

³In applied and atheoretical contexts, a similar equation has been extensively used link trade to income, distance and other characteristics of countries. It proved to be successful in capturing the actual patterns of trade. Here, we will look at these results in light of the predictions of the model. The simplest empirical gravity equation regresses the volume of trade between bilateral pairs of countries (regions) on their bilateral distance, income, and various dummy variable (common border, language, etc...). It works really well in terms of fitting the data. However, since such simple gravity equation is different from the one derived from our model — and in principle we would like to use it to perform counterfactual experiments, the first order task is to understand the theory behind it first.

lowing equation:

$$\frac{X_{ni}}{X_n} = \frac{X_i}{\sum_n X_n} \left(\frac{d_{ni}}{P_i P_n} \right)^{1-\sigma}, \quad (1.7)$$

where $X_{ni} = p_{ni}c_{ni}$ are expenditures of country n on goods from country i , $X_n = \sum_i X_{ni}$ are total expenditures of country i on all goods (equal to country n 's income $w_n L_n$), and $P_i = (\sum_i p_{ni}^{1-\sigma})^{\frac{1}{1-\sigma}}$ is the ideal CPI price index (price level weighted by the actual consumption share of each good).

Proof. Note that the household's problem can be summarized by the following Lagrangian:

$$\mathcal{L}_n = \left(\sum_{i=1..N} \alpha_i^{\frac{1}{\sigma}} c_{ni}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} - \frac{1}{P_n} \left(\sum_{i=1..N} p_{ni}c_{ni} - w_n \right), \quad (1.8)$$

where by definition of the Lagrange multiplier, $\frac{1}{P_n}$ is the shadow value of one unit of income in terms of the composite consumption U_n , and P_n is the shadow price of a unit of composite consumption U_n . Using the order conditions to this problem,

$$\frac{\partial \mathcal{L}_n}{\partial c_{ni}} = U_n^{\frac{1}{\sigma}} \alpha_i^{\frac{1}{\sigma}} c_{ni}^{\frac{-1}{\sigma}} - \frac{p_{ni}}{P_n} = 0, \quad \text{all } i \quad (1.9)$$

It is easy to link this multiplier to prices:

$$\begin{aligned} P_n^{\sigma-1} \alpha_i p_{ni}^{1-\sigma} &= U_n^{\frac{1-\sigma}{\sigma}} \alpha_i^{\frac{1}{\sigma}} c_{ni}^{\frac{\sigma-1}{\sigma}} \\ P_n^{\sigma-1} \sum_i \alpha_i p_{ni}^{1-\sigma} &= U_n^{\frac{1-\sigma}{\sigma}} \sum_i \alpha_i^{\frac{1}{\sigma}} c_{ni}^{\frac{\sigma-1}{\sigma}} \\ P_n^{\sigma-1} \sum_i \alpha_i p_{ni}^{1-\sigma} &= \left(\sum_i \alpha_i^{\frac{1}{\sigma}} c_{ni}^{\frac{\sigma-1}{\sigma}} \right)^{-1} \sum_i \alpha_i^{\frac{1}{\sigma}} c_{ni}^{\frac{\sigma-1}{\sigma}}, \\ P_n &= \left(\sum_i \alpha_i p_{ni}^{1-\sigma} \right)^{\frac{1}{1-\sigma}}. \end{aligned} \quad (1.10)$$

We refer to it as an ideal price index.

Next, from the first order conditions

$$\begin{aligned}\frac{p_{ni}}{P_n} &= U_n^{\frac{1}{\sigma}} \alpha_i^{\frac{1}{\sigma}} c_{ni}^{\frac{-1}{\sigma}}, \\ \frac{p_{nj}}{P_n} &= U_n^{\frac{1}{\sigma}} \alpha_j^{\frac{1}{\sigma}} c_{nj}^{\frac{-1}{\sigma}},\end{aligned}$$

and the definition of expenditures $X_{ni} \equiv p_{ni}c_{ni}$, we derive

$$\frac{X_{nj}}{X_{ni}} = \frac{\alpha_j}{\alpha_i} \left(\frac{p_{nj}}{p_{ni}} \right)^{1-\sigma}.$$

Summing up the above expression wrt j , we obtain:⁴

$$\frac{X_{ni}}{X_n} = \alpha_i \left(\frac{p_{ni}}{P_n} \right)^{1-\sigma}. \quad (1.11)$$

Multiplying both sides of equation (1.11) by X_n , and summing up wrt n , we use the law of one price

$$p_{ni} = d_{ni}p_{ii}, \quad (1.12)$$

and the balanced trade condition (implied by (1.2))

$$X_i = Y_i = \sum_n X_{ni}, \quad (1.13)$$

to obtain

$$\alpha_i p_{ii}^{1-\sigma} = \frac{X_i}{\sum_n \left(\frac{d_{ni}}{P_n} \right)^{1-\sigma} X_n}. \quad (1.14)$$

⁴The demand for each good i in country n is given by

$$c_{ni} = \alpha_i \left(\frac{p_{ni}}{P_n} \right)^{-\sigma} \left(\frac{X_n}{P_n} \right).$$

The above equation links prices (1.11) to aggregate variables. We use it to substitute for prices in (1.11) (after using (1.12)), and derive:

$$X_{ni} = \alpha_i p_{ii}^{1-\sigma} \left(\frac{d_{ni}}{P_n} \right)^{1-\sigma} X_n = \frac{X_n X_i}{\sum_n \left(\frac{d_{ni}}{P_n} \right)^{1-\sigma} X_n} \left(\frac{d_{ni}}{P_n} \right)^{1-\sigma}. \quad (1.15)$$

In addition, in the special case when the iceberg transportation cost ($d_{ni} = d_{in}$) is symmetric, we can show that

$$\sum_n \left(\frac{d_{ni}}{P_n} \right)^{1-\sigma} X_n = P_n^{1-\sigma} \sum_i X_i, \quad (1.16)$$

and instead of (1.15) obtain an even simpler expression (cumbersome to derive):

$$X_{ni} = \frac{X_n X_i}{\sum_n X_n} \left(\frac{d_{ni}}{P_i P_n} \right)^{1-\sigma}.$$

■

Exercise 5 Consider the following expenditure minimization problem:

$$E(U) = \min_{(c_i)_{1..N} \geq 0} \sum_{i=1}^N p_i c_i$$

subject to

$$\sum_{i=1}^N (\alpha_i^{\frac{1}{\sigma}} c_i^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} = U, \\ c_i \geq 0, \text{ all } i = 1..N,$$

where p_i 's denote prices, $E(U)$ are total expenditures (given U), α_i 's are the preference weights, σ is the elasticity of substitution, and U is the 'composite good' consumption level (or simply utility). Assume that p_i 's, α_i 's, σ and U are all strictly positive.

a. Show that $E(U)$ is homogenous of degree 1 ($E(\mu U) = \mu E(U)$, all $\mu > 0$), and thus

takes the form $P \times U$ where $P = E(1)$.

b. Prove the Envelope Theorem in the context of the problem stated in (a), i.e. show that $E'(U) = \lambda$, where λ is the Lagrange multiplier on the constraint in (a). Then, use the conclusion from point (a) to say $E'(U) = P$, and thus by Envelope Theorem to say $\lambda = P$. Using it, solve for $E(1)$, which together with (a) shows

$$E(U) = \left(\sum_{i=1}^N \alpha_i p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}} U.$$

(This is an alternative way of deriving the price index to the one we did in the proof of the proposition above.)

c. Show that the expenditures minimization problem with $E(U) = \text{Income}$, is equivalent the underlying utility maximization problem given by:

$$U = \max_{c_i \geq 0} \left(\sum_{i=1..N} \alpha_i^{\frac{1}{\sigma}} c_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}},$$

subject to

$$\sum_{i=1..N} p_i c_i = \text{Income}.$$

Exercise 6 Suppose that the preferences of the household are instead described by:

$$U_n = (C^{NT})^\gamma \left(\sum_{i=1..N} \alpha_i^{\frac{1}{\sigma}} c_{ni}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{(1-\gamma)\sigma}{\sigma-1}},$$

where C^{NT} is the consumption of the local non-tradable good (services). Assume that production technology of the non-tradable good is linear and assume that labor is perfectly mobile across the two sectors. In the extended model, derive the gravity equation by modifying each step in the above proof accordingly. *HINT: Use the fact that this is a Cobb-Douglas aggregator, and so it implies that non-tradable goods have a constant share in the overall consumer expenditures. You should get exactly the same gravity equation with total expenditures replaced by total expenditures on all*

tradable goods.

The existence of the first few terms in equation (1.7) should be intuitive. In fact, we should expect the share of expenditures on good i in total expenditures of country n are positively related to the size of country i (measured by income or labor endowment), and negatively related to the bilateral trade barrier d_{ni} between them—with the strength of the latter effect depending on the elasticity of substitution σ . However, there are more terms in the gravity equation. Trade flows turns out to additionally depend on the endogenous product of price indices of the two countries $P_n P_i$ — a term referred to by Anderson and Wincoop as ‘gravitas’. Our next task is to link this term to the primitives in the model.

Gravity with Gravitas

Let’s first take a look at the formula for the price level in country n ,

$$P_n = \left(\sum_{i=1..N} \alpha_i (d_{ni} p_{ii})^{1-\sigma} \right)^{\frac{1}{1-\sigma}}.$$

and think what makes a country price level high. Since all countries face the same p_{ii} ’s, we observe that high P can arise as consequence of: (i) high overall level of d_{ni} ’s, and/or (ii) high positive correlation of d_{ni} ’s with p_{ii} ’s. Thus, if we think of the iceberg cost d_{ni} ’s in terms of distance between countries in some space, (i) means that a country is distant from all other countries, and (ii) means that a country is distant from the countries that are least distant from the rest of the world⁵. Clearly, both (i) and (ii) are an indication of isolation.

Figure 1.1 illustrates an example of such situation, which will naturally arise when we are dealing with regions of a large country and regions of a small country. An

⁵Because the price p_{ii} is high, the good produced by the country must be in high demand. This happens when the country is close to all the other countries (rest of the world).

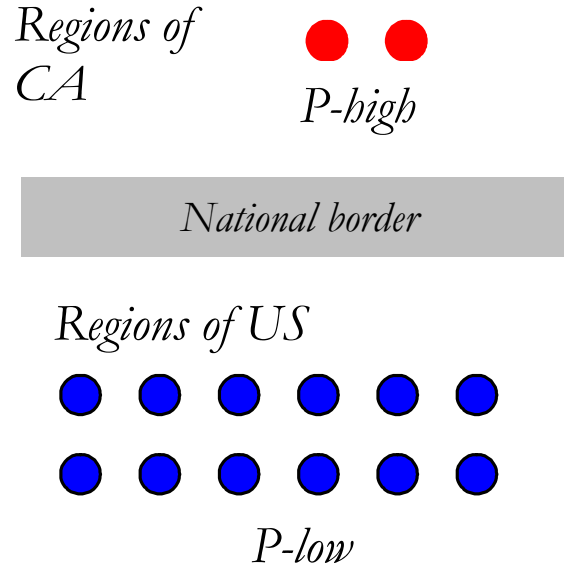


Figure 1.1: An example of isolated small country, and implications for aggregate price level.

obvious example would be the case of the states of the US (large country), and the provinces of Canada (small country).

Upon closer inspection of equation (1.7), we note the following:

- Observation 1: The multilateral resistance term ($\frac{1}{P_i P_n}$) in the gravity equation makes the small country (two isolated dots in Figure 1) to trade relatively more with each other.
- Observation 2: The multilateral resistance term makes the large country (dots that are not isolated in Figure 1) to trade relatively more with the small country.

Formally, we can derive the above two observations as follows. For the sake of argument, let's simply denote the two isolated regions provinces of Canada, and rest of the regions the states of the US (d stands for the cost of crossing the national border), and set the following notation for their underlying price levels: $P_{CA} = high$,

$P_{US} = low$. Simplifying also the notation for iceberg transportations, $d_{CA,US} = d > 0$, $d_{CA,CA} = d_{US,US} = 1$, we obtain from gravity equation:

$$\begin{aligned}\frac{X_{CA,US}}{X_{CA,CA}} &= \frac{\frac{X_{CA,US}}{X_{CA}}}{\frac{X_{CA,CA}}{X_{CA}}} = \frac{X_{US} \left(\frac{d}{high \times low} \right)^{1-\sigma}}{X_{CA} \left(\frac{1}{high \times high} \right)^{1-\sigma}} = \frac{X_{US}}{X_{CA}} d^{1-\sigma} \left(\frac{low}{high} \right)^{\sigma-1}, \\ \frac{X_{US,CA}}{X_{US,US}} &= \frac{\frac{X_{US,CA}}{X_{US}}}{\frac{X_{US,US}}{X_{US}}} = \frac{X_{CA} \left(\frac{d}{high \times low} \right)^{1-\sigma}}{X_{US} \left(\frac{1}{low \times low} \right)^{1-\sigma}} = \frac{X_{CA}}{X_{US}} d^{1-\sigma} \left(\frac{high}{low} \right)^{\sigma-1}.\end{aligned}$$

As we can see, the additional endogenous term $\left(\frac{low}{high} \right)^{\sigma-1}$ does make $\frac{X_{CA,US}}{X_{CA,CA}}$ higher and it does make $\frac{X_{US,CA}}{X_{US,US}}$ lower as claimed above.

The Economics Behind Gravitats As we explain below, in the context of the example illustrated in Figure 1.1, the key feature that the asymmetric size between the small country and the large country is that the demand for imported goods is more elastic in the large country than in the small country, and that the supply of foreign goods is more elastic in the small country than in the large country.

To see this conclusion, simply note that the households from the larger country can more effectively shift their expenditures from the foreign goods towards the domestic goods. For instance, in the context of the example considered above (Figure 1), when the US consumers cut spending on each Canadian good by \$1, they must shift only \$2 of spending on 12 home goods. However, when the Canadian consumers cut spending by \$1 on each US good, they must shift as much as \$12 on only 2 Canadian goods. Now, because the marginal utility from consumption of each good declines, the immediate consequence of this property is a more elastic demand for Canadian goods in the US than the demand for American goods in Canada.⁶ (The opposite conclusion applies to the supply side because the US producers face a much smaller

⁶In simple words, when all Canadian goods become more expensive in the US, the US households can shift to a wide variety of home goods, but if all American goods become more expensive in Canada, Canadians have to take the hit.

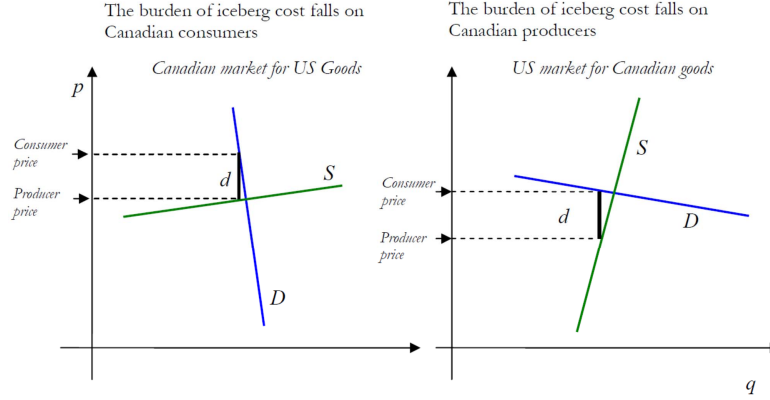


Figure 1.2: Canadian Producers and Consumers Pay the Iceberg Cost.

decline in the price as move their sales from Canada (i.e. one unit from each province) to US than the Canadian producers face as they move their sales from the states (i.e. one unit from each state) to Canada.)

Exercise 7 To formalize the above argument, solve for the demand from the following problem:

$$\max(q_A^{\frac{\sigma-1}{\sigma}} + Nq_B^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}}$$

subject to

$$p_A q_A + N p_B q_B = I.$$

Specifically, derive the demand for good B, and show that for large N the price index will be affected by the price p_B – implying a lower measured price elasticity of demand. HINT: Derive an equation analogous to (1.11). Calculate the price index when N is infinite.

Given the described above implication of relative size on elasticities, it should

not surprise that an increase in the iceberg transportation cost d might have a very different effect on the two countries. These elasticities determine who bears the burden of this cost. As illustrated in Figure 1.2, in this case these are the consumers and producers of the small isolated country who will pay for it. Thus, isolation implies that the terms of trade (price of imports in terms of exports) of the small country worsens relative to the large country, and the worsened terms of trade makes the small country shift relatively more spending towards the home goods than the large country. This is the economic intuition why the two endogenous terms that appear in the theoretical gravity equation convey, and it is a beautiful example how general equilibrium considerations sometimes matter.

To formalized the above idea, let's push the previously used argument to the limit and make US arbitrarily large relative to Canada (the inelastic demand and supply lines become vertical). In such case, assuming each region of the same size normalized to 1 ($L_i = L_j = 1$), as number of regions in the large country goes to infinity, we have

$$\frac{p_{CA,CA}}{p_{US,US}} = \frac{P_{CA}}{P_{US}} = d,$$

and thus

$$\begin{aligned} \frac{X_{CA,US}}{X_{CA,CA}} &= \frac{X_{US}}{X_{CA}} d^{1-\sigma} \left(\frac{P_{US}}{P_{CA}} \right)^{\sigma-1} = \frac{p_{US,US}}{p_{CA,CA}} d^{1-\sigma} \left(\frac{P_{US}}{P_{CA}} \right)^{\sigma-1} = d^{-1} d^{1-\sigma} d^{1-\sigma} = d^{1-2\sigma}, \\ \frac{X_{US,CA}}{X_{US,US}} &= \frac{\frac{X_{US,CA}}{X_{US}}}{\frac{X_{US,US}}{X_{US}}} = \frac{X_{CA} \left(\frac{d}{\text{high} \times \text{low}} \right)^{1-\sigma}}{X_{US} \left(\frac{1}{\text{low} \times \text{low}} \right)^{1-\sigma}} = d d^{1-\sigma} d^{\sigma-1} = d. \end{aligned}$$

If, however, we did not have the 'gravitas', we would have obtained instead:

$$\begin{aligned} \frac{X_{CA,US}}{X_{CA,CA}} &= \frac{X_{US}}{X_{CA}} d^{1-\sigma} = \frac{p_{US,US}}{p_{CA,CA}} d^{1-\sigma} = d^{-1} d^{1-\sigma} = d^{-\sigma}, \\ \frac{X_{US,CA}}{X_{US,US}} &= \frac{\frac{X_{US,CA}}{X_{US}}}{\frac{X_{US,US}}{X_{US}}} = \frac{X_{CA} \left(\frac{d}{\text{high} \times \text{low}} \right)^{1-\sigma}}{X_{US} \left(\frac{1}{\text{low} \times \text{low}} \right)^{1-\sigma}} = d d^{1-\sigma} = d^{2-\sigma}. \end{aligned}$$

Does the Model Fit the Data?

The simplest test of the model is to look at the predicted asymmetric effect of trade costs on trade between a small country like Canada and a large country like US (that are somewhat isolated from the rest of the world). In such case, the Armington model predicts that if crossing the national border involves a cost (tariff and non-tariff barriers), then the impact of this cost should be asymmetric. In particular, such cost should drastically reduce trade of each Canadian province with the US, but should not reduce as much the trade of each US state with the Canadian provinces. By running 2 simple regressions, we can check if this is the case.

The empirical specification that we are going to adopt will simply assume that the iceberg cost of transportation is a function of distance and the national border. We will do the same for gravitas. Formally, we are going to have: $d_{ni} = \exp(\tau b_{ni}) \delta_{ni}^\rho$, $P_n P_i = \exp(v b_{ni})$, where b_{ni} a border dummy (1 if there is national borders between regions n and i , 0 otherwise), and δ_{ni} is the distance between n and i , τ is border cost affecting bilateral trade barrier, and v is the border effect operating through the multilateral resistance term.

Given the specification of the iceberg cost and gravitas, plugging into (1.7), we thus need to estimate the following equation:

$$\log X_{ni} = \kappa + A \log X_n + B \log X_i + C \log \delta_{ni} + D b_{ni},$$

where $C = \rho(1 - \sigma)$ and $D = (1 - \sigma)(\tau + v)$.

The equation takes into account the multilateral resistance term $P_i P_n$ in the form of a border dummy. Based on our previous analysis, we should expect this term to capture well the notion of isolation when regressed from the US side and the Canadian side separately. In particular, based on our discussion, we should expect to find that D is much higher when we run the regression from the Canadian side (CA-CA and

Table 1.1: Comparing CA-US and US-CA gravity equations.

| Parameter | Regression from CA side | Regression from US side |
|-----------|-------------------------|-------------------------|
| κ | 2.80 (.12) | .41 (.05) |
| A | 1.22 (.04) | 1.13 (.03) |
| B | .98 (.03) | .98 (.02) |
| C | -1.38 (.07) | -1.08 (.04) |
| D | -16.4 (2.0) | -1.5 (.08) |
| R^2 | .76 | .85 |

CA-US observations), than when we run it from the US side (US-US and US-CA observations). The results are as follows (Replicated Table 1 from Anderson and Wincoop, 2003):

Exercise 8 *Go to Anderson's website. Download the zip file with the dataset supporting the paper. Replicate the above regressions using this dataset.*

As we can see, the estimated values do exhibit strong asymmetry. In fact, D is by far more negative in the regression from the Canadian side than in the regression from the US side. Each Canadian province, controlling for income and distance, trades 1600% more with another Canadian province than with a US state. Given such huge asymmetry in the regression, our next question should be whether the model is quantitatively capable of generating it.

Exercise 9 *(Numerical experiment with the model) Consider the Armington model with the following parameter setting: $N = 100$, $\alpha = 1/N$, $\sigma = 11$, $L_i = L_j = 1$, all $i, j = 1..N$. Assume that the first 90 of the N regions are in a large country (US), and the last 10 are in a small country (Canada), which roughly corresponds to the ratio of Canadian GDP to the US GDP. Furthermore, assume that the transportation cost between the regions within the same country is zero, i.e. $d_{ni} = 1$ whenever $i, n \in US$, or $i, n \in CA$, and assume that the iceberg transportation cost between the regions*

within two different countries is 20%, i.e. $d_{ni} = 1.2$ whenever $i \in US$, $n \in CA$ or $i \in CA$, $n \in US$.

a. Use the following equilibrium relation from the model

$$p_{ii}L_i = \sum_n X_{ni} = \sum_n \alpha_i \left(\frac{X_{ni}}{X_n} \right) X_n = \sum_n \alpha_i \left(\frac{d_{ni}p_{ii}}{P_n} \right)^{1-\sigma} p_{nn}L_n$$

to construct an iterative algorithm that solves the model in MATLAB.⁷ Using the algorithm, compute the overall price level of a representative US region and Canadian region, and the prices of the corresponding goods. (HINT: The algorithm may be unstable unless you slow down the updates a bit. To be on the safe side, I suggest to divide both sides by $L_i p_{ii}^{1-\sigma}$, compute p_{ii} , and use the updating rule that puts .5 weight on the old value and only .5 weight on the newly solved value.⁸ $p_{i+1} = .1p' + .9p_i$, where i is the iteration number, and p_i is used to solve for the vector p' in iteration i . Don't forget to evaluate the convergence, and the residuals of equilibrium conditions at the end. Remember that p_{NN} is the numeraire. (Print out the code and hand in with the HW.)

b. What is the home-bias from the US side (defined as $X_{US,US}/X_{US,CAN}$) and from the Canadian side (defined as $X_{CA,CA}/X_{CA,US}$).

c. Using data generated by the model, suppose you run the following regression of trade flows on the border dummy (referred to as the 'McCallum regression'):

$$\ln \frac{X_{ni}}{X_i X_n} = \kappa + A \times \text{border_dummy} + \varepsilon,$$

where $n \in CA$, $i \in CA$ or US . What is the value of the regression coefficient on the border dummy?

d. Suppose you run the same regression as in point c but from the US side, i.e.

⁷If it is a contraction, then it will converge to the fixed point.

⁸This way you enlarge the domain on which out mapping is contraction. You then do not need a very precise guess for convergence to the fixed point to occur.

$n \in US$, $i \in CA$ or US . What is the value of the regression coefficient on the border dummy?

e. How do your answers to c and d compare to the coefficients that Anderson and Wincoop found in the data by running McCallum's regression separately from the US side and the Canadian side? Explain briefly the implications of your findings.

f. Redo points c and d with $\sigma = 8$.

g. Comparing the answers in e and f, what fraction of the border effect is accounted for by the endogenous multilateral resistance term?

h. What is the average share of trade with the US for a representative Canadian province in the model (measure it by $(90X_{CA,US})/(10X_{CA,CA} + 90X_{CA,US})$)? Consider two levels of trade cost: $d_{US,CA} = 1.2$ (same as before), and $d_{US,CA} = 1.175$. Given that the median and average value of this object in the data is about⁹ .45, which level of the border cost accounts better for this number?

i. Would the answers to b-g change if instead you had 100 Canadian regions and 900 US regions? Explain your answer analytically.

Structural Estimation

Anderson and Wincoop (2003) structurally estimate the model using a set of 10 provinces, 30 states of US and 20 OECD countries. Their exercise is meant to address the question whether the model can quantitatively fit the data for plausible parameter values. An alternative approach to theirs would be a detailed calibration of the model in the spirit of the numerical example you solved above.

Specification To structurally estimate the model, Anderson and Wincoop use the following specification for the iceberg transportation cost:

$$d_{ni} = \exp(\beta b_{ni}) \times \delta_{ni}^{\rho}, \quad (1.17)$$

⁹Pulled out from the data available from Anderson's website.

where b_{ni} is the national border dummy (1 if there is a national border between region n and region i , 0 otherwise), δ is the distance between regions n and i in miles, ρ is the impact parameter of distance between on the implied iceberg transportation cost, and β is the impact parameter of the national border on the implied iceberg transportation cost.

Substituting out d_{ni} in the theoretical gravity equation, they obtain the following empirical specification of the model:

$$\ln \frac{X_{ni}}{Y_i Y_n} = k + a_1 \ln(\delta_{ni}) + a_2 b_{ni} - \ln P_i^{1-\sigma} - \ln P_n^{1-\sigma} + \varepsilon_{ni}, \quad (1.18)$$

where $a_1 = (1 - \sigma)\rho$, $a_2 = (1 - \sigma)\beta$, and the vector of aggregate prices $(P_n)_n$ solves to the fixed point problem given by (1.16):¹⁰

$$\begin{aligned} P_n^{1-\sigma} &= \sum_{i=1, \dots, N} \frac{\exp(a_1 \ln(\delta_{ni}) + a_2 b_{ni})}{P_i^{\sigma-1}} X_i, \quad n = 1, \dots, N-1, \\ P_N &= 1. \end{aligned} \quad (1.19)$$

Note that the observable data includes distance matrix $(\delta_{ni})_{ni}$, border dummy matrix $(b_{ni})_{ni}$, multilateral expenditure shares $(\frac{X_{ni}}{Y_i Y_n})_{ni}$, and income vector $(Y_i)_i$. The price vector $(P_n)_n$ is unobservable, and so we must use theory to solve for it.

Numerical Algorithm to Estimate the Model (1.18)

- Set $\sigma = 6$ in consistency with the estimates of the long-run impact of a change in tariff rates on trade from the literature.¹¹
- Set the values of a_1, a_2 , and solve for the vector of prices $(P_n)_n$ from the fixed point problem given by (1.19).

¹⁰Income data is assumed to be normalized so that $\sum_n X_n = 1$.

¹¹Note that σ can not be identified separately from the other parameters.

- Plug in the price vector $(P_n)_n$ into the regression equation (1.18), and find the constant κ that minimizes the squared sum of regression residuals. Given residual minimizing value of κ , evaluate the squared sum of residuals $r = \sum_{ni} \varepsilon_{ni}^2$.
- Repeat steps 2-4 above by choosing a_1, a_2 to minimize the residual r calculated in step 3.

Results The results of estimating the structural model are presented in the table reproduced below (Table 2 in the paper). As we can see, the model does an magnificent job in account for the asymmetry and the border puzzle. In the two country case (second column of Table 2), it underpredicts trade between Canadian province on average by only 17%, and overpredicts trade of US states with other US states by 6%. Given that Canadian provinces trade 1600% more with each other than with US states, this is a huge success. In addition, Anderson and Wincoop show that when the model is extended to include other countries, it does an even better job (see last column in Table 2 in the paper).

A Note on the Literature

This part was based on two influential papers: McCallum (1995) and Anderson and Wincoop (2003). McCallum's paper shows that an ad hoc gravity equation on trade between Canada and US (from Canadian side) yields puzzling results. Namely, after controlling for distance and income, Canadian provinces trade 2200% more with another Canadian province than with US state. The original paper interprets this finding as possibly suggesting an enormous cost of crossing the border, and is referred to as 'the border puzzle'. Anderson and Wincoop (2003) is a response to this finding. Anderson and Wincoop show that according to the theory, the specification of the ad hoc gravity model in most applications is incomplete, and so the results may be biased

1.4 Dornbusch, Fisher and Samuelson Model

Dornbusch, Fisher and Samuelson model is the most general version of the Ricardian model (DFS model hereafter) for the case of two countries. The key idea is to span goods on a unit interval, and thus summarize the endogenous equilibrium specialization pattern by two cutoff values (pivotal goods) defining *the set of goods that are produced only by country 1 and the set of goods that are produced only by country 2*.

The DFS model nests a two country Armington model, it also nests the two country Eaton and Kortum model discussed in the next section. Interestingly, the DFS model is particularly difficult to extend to a multicountry framework in full generality, and it wasn't until Eaton and Kortum parameterization that this framework took off as a basis for any quantitative analysis.

The exercise below will walk you through a simple symmetric version of the DFS model. In particular, you will establish here its relation to the Armington model, and solve for the cutoff values. Later, we will find all these results useful to understand the intuition behind the Eaton and Kortum (2002) model.

Exercise 10 (*Dornbusch, Fisher and Samuelson (1977)*) Consider a world with two **symmetric** countries and a continuum of goods indexed on a unit interval. Preferences in each country are identical and given by

$$U_i = \left(\int_0^1 \ln c_i(\omega) d\omega \right), \quad i = 1, 2,$$

and all markets are perfectly competitive. Assume each country has access to a linear technology to produce each good using labor,

$$y_i(\omega) = z_i(\omega)l_i(\omega),$$

where $z_i(\omega)$ is the efficiency level in producing good ω in country i , and $l_i(\omega)$ is the labor input. Assume that the labor endowment of the stand-in household in each country is one, and the production efficiency schedules are given by the following functions:

$$\begin{aligned} z_1(\omega) &= e^{1-\omega}, \\ z_2(\omega) &= e^{\omega}. \end{aligned} \tag{1.20}$$

In addition, assume there is a positive tariff rate T between the two countries that amounts to 10% of the value of the transported goods across the border. The revenue from the tariff is lump-sum rebated to the households.

- a. Define competitive equilibrium for this economy.
- b. **Refers to point a above.** Compute the competitive equilibrium you have defined in a. *HINT: Find 2 cutoffs that divide the space of goods into 3 categories: (i) traded and produced in country 1, (ii) traded and produced in country 2, and (iii) not traded (both countries produce them for home market only). Exploit symmetry to say that wages must be 1 in both countries. Use the fact that in the case of log utility the share of expenditures on each good is always a constant fraction of total expenditures on all goods.*
- c. Apply NIPA rules to compute the GDP of each economy. What happens to the GDP in equilibrium when the tariffs are increased? *HINT: Read handbook of NIPA accounting available from BEA website.*¹²
- d. How would you have to modify the assumed efficiency schedules stated in (1.20) to effectively obtain a symmetric two-country Armington model. Based on your answer, what is the key qualitative difference between the Armington model and the DFS model.

¹²See <http://www.bea.gov/national/pdf/NIPAhandbookch1-4.pdf>.

1.5 Hecksher-Ohlin Model

The following exercise will walk you through the setup of the 2x2 Hecksher-Ohlin Model. In this version of the Hecksher-Ohlin model countries have access to the same technologies to produce 2 goods, but differ in factor endowment of capital and labor. Because technologies to produce each good use these two factor at different intensities, in equilibrium countries partially specialize in the production of the good more intensive in the abundant factor. The specialization leads to a very peculiar result: Despite the fact that factors are immobile across countries, trade in goods leads to *factor price equalization* across countries (wages and interest rates are the same).¹³

Exercise 11 (*Hecksher-Ohlin model*) Consider the world with 2 countries and 2 tradable goods. Preferences of the stand-in household in each country are

$$U_i = \sum_{j=1,2} \log C_i^j, \quad i = 1, 2$$

where C_i^j denotes consumption in country i of good j . The stand-in household in country 1 has 2 units of labor (L) and 3 units of capital (K), and the stand-in household in country 2 has 3 units of labor and 2 units of capital. Firms in each country have access to the same CRS technology to produce both goods. The technology to produce good 1 is $Y = K^{\frac{1}{3}}L^{\frac{2}{3}}$ (sector 1), and good 2 is $Y = K^{\frac{2}{3}}L^{\frac{1}{3}}$ (sector 2). For simplicity assume there is no transportation cost.

- a. Assume factors are perfectly mobile across countries. Define the competitive equilibrium.
- b. **Refers to equilibrium defined in a.** It can be shown, using the First and the Second Welfare Theorems, that the competitive equilibrium is unique up to the **unde-**

¹³These two results are referred to as the Hecksher-Ohlin Theorem and the Factor Price Equalization Theorem.

terminated allocation of capital and labor across countries within sectors, and it solves the following planning problem for $\mu = \frac{1}{2}$:

$$\max_{(C_i^j, K_i^j, L_i^j)_{i,j=1,2}} \mu \sum_{j=1,2} \log C_1^j + (1 - \mu) \sum_{j=1,2} \log C_2^j$$

subject to

$$\begin{aligned} \sum_{i,j=1,2} K_i^j &= 5, & \sum_{i,j=1,2} L_i^j &= 5, \\ \sum_{i=1,2} C_i^1 &= \sum_i (K_i^1)^{\frac{1}{3}} (L_i^1)^{\frac{2}{3}}, \\ \sum_{i=1,2} C_i^2 &= \sum_i (K_i^2)^{\frac{2}{3}} (L_i^2)^{\frac{1}{3}}. \end{aligned}$$

Compute the competitive equilibrium you defined in a. (HINT: Remember that allocation is undetermined wrt to allocation of production across countries (who produces what). Exploit symmetry to argue that the relative price between goods and factors must be 1. Then, introduce an aggregate firm that produces the entire world output (max output), and compute K, L from its problem.)

c. Assume factors are immobile across countries. Define the competitive equilibrium.

*d. **Refers to equilibrium defined in c.** It can be shown that there is a unique competitive equilibrium, and it solves the following planning problem for $\mu = \frac{1}{2}$:*

$$\max_{(C_i^j, K_i^j, L_i^j)_{i,j=1,2}} \mu \sum_{j=1,2} \log C_1^j + (1 - \mu) \sum_{j=1,2} \log C_2^j$$

subject to

$$\begin{aligned}
 (*) \quad \sum_{j=1,2} K_1^j &= 3, \quad \sum_{i,j=1,2} L_1^j = 2, \quad \sum_{j=1,2} K_2^j = 2, \quad \sum_{j=1,2} L_2^j = 3, \\
 \sum_{i,j=1,2} K_i^j &= 5, \quad \sum_{i,j=1,2} L_i^j = 5, \\
 \sum_{i=1,2} C_i^1 &= \sum_i (K_i^1)^{\frac{1}{3}} (L_i^1)^{\frac{2}{3}}, \\
 \sum_{i=1,2} C_i^2 &= \sum_i (K_i^2)^{\frac{2}{3}} (L_i^2)^{\frac{1}{3}}.
 \end{aligned}$$

Compute the competitive equilibrium you defined in point c. (**HINT:** Guess that the solution to the planning problem above solves a relaxed problem with (*) constraints omitted (like in the planning problem in point b). Verify the guess by showing that factor markets clear – use (*) in combination with the factor demand functions you derived in problem 2 to find market clearing production pattern (write it in matrix form, will be easier...).

e. What is the pattern of trade in the competitive equilibrium you found in d? More precisely, in which good the labor abundant country is a net exporter?

f. Note that ‘the trick’ you used in d to solve for the equilibrium would not work in general, i.e. for an arbitrary distribution of factor endowment levels¹⁴. Show which step of your solution in e would break down if this was not true, and explain why.

HINT: Recall that there are non-negativity constraints on all the variables.

1.6 Eaton and Kortum Model

Essentially, Eaton and Kortum (2002) model is a versatile and tractable probabilistic parameterization of the Ricardian model with a continuum of goods due to Dornbusch, Fisher and Samelson (1977). EK model extends the DFS framework to

¹⁴The range of endowment vectors for which ‘the trick’ works is referred to as the cone of diversification.

a multicountry context, and allows for an explicit derivation of the gravity equation.

Model Economy

Goods are indexed on a unit interval $\omega \in [0, 1]$, and the world is comprised of N countries (regions). Every country can produce every good from the continuum, but the labor requirement to produce each good differs. Unlike in the DFS model, the productivity schedules are described probabilistically. Namely, it is assumed that the efficiency of producing a good in country n is a realization of an i.i.d. Frechet distributed random variable \mathcal{Z}_n :

$$F_n(\mathcal{Z}_n \leq z) = \exp(-T_n z^{-\theta}), \quad (1.21)$$

where T_n and θ are parameters governing the mean and the dispersion¹⁵, and n is the country index.

As before, geography is modeled by an iceberg transportation cost obeying three standard properties: (i) symmetry $d_{ni} = d_{in} \geq 1$, (ii) no cost within the country $d_{ii} = 1$, and (iii) the triangle inequality

$$d_{ni} \leq d_{nj} + d_{ji} \quad \text{all } i, j, n = 1..N. \quad (1.22)$$

Probabilistic Notion of Comparative Advantage

Figure 1.3 illustrates the plots of the Frechet density function. The moments of this distribution are given by: (i) mean= $T^{\frac{1}{\theta}} \Gamma(1 - \frac{1}{\theta})$, (ii) coefficient of variation (standard deviation/mean) = $\exp(\frac{\pi}{\theta\sqrt{6}})$. Because θ unambiguously determines the coefficient of variation, the two parameters have a natural interpretation: (i) T_i char-

¹⁵We will use the convention of denoting a random variable by a caligraphic capital letter. By the law of large numbers, note that $F(z)$ is also the fraction of goods produced at efficiency z or lower.

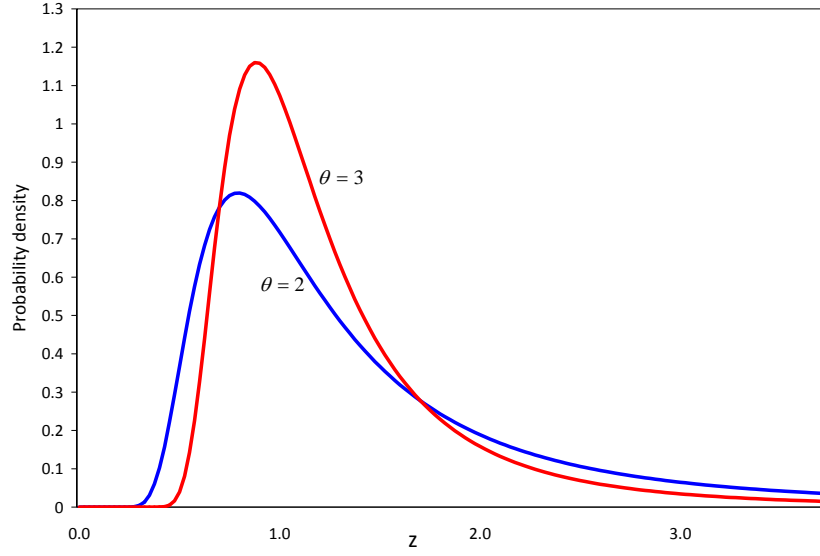


Figure 1.3: Frechet density function.

acterizes the overall level of technology of a country (absolute advantage), and (ii) θ , a parameter common to all countries, characterizes the dispersion of efficiency across goods (comparative advantage).

Frechet distribution, or in general any exponential distribution, has the following *four* properties that will greatly simplify our analysis of the model:

- *Property 1 (Frechet distributed extreme values):* Let $(Z_1)_{i=1..N}$ be a vector of Frechet distributed random variables with parameters $(T_i, \theta_i)_i$. Then,

$$Z = \max_i \{Z_i\}, \quad (1.23)$$

is Frechet distributed with parameter $T = \sum_i T_i$, and θ .

- *Property 2 (Mean determined winning probability):* Let $(Z_i)_{i=1..N}$ be a vector

of Frechet distributed random variables with parameters $(T_i)_i, \theta$. Then,

$$\Pr(\mathcal{Z}_s \geq \max_{i \neq s} \{\mathcal{Z}_i\}) = \frac{T_s}{\sum_i T_i}. \quad (1.24)$$

- *Property 3 (Memorylessness):* Let \mathcal{Z} be a Frechet distributed random variables with parameters (T, θ) . Then, the conditional distribution is equal to the unconditional distribution

$$\Pr(\mathcal{Z}_i \leq z_2 | \mathcal{Z} \leq z_1) = e^{-T z_2^{-\theta}}.$$

- *Property 4 (Scale invariant dispersion):* Let \mathcal{Z} be a Frechet distributed random variables with parameters (T, θ) . Then, the distribution of a random variable $a\mathcal{Z}$ ($a \in R_+$) is Frechet with parameters $(a^{-\theta}T, \theta)$.

Proof. Property 1:

$$\Pr(\max_i \{\mathcal{Z}_i\} \leq z) = \prod_i \Pr(\mathcal{Z}_i \leq z) = e^{-\sum_i T_i z^{-\theta_i}}$$

Property 2:

$$\begin{aligned} \Pr(\mathcal{Z}_s \geq \max_{i \neq s} \{\mathcal{Z}_i\}) &= \\ \int_0^\infty \prod_i \Pr(\mathcal{Z}_i \leq z_s) dF(z_s) &= \int_0^\infty \prod_i \Pr(\mathcal{Z}_i \leq z_s) F'(z_s) dz_s = \\ &= \int_0^\infty \theta z_s^{-\theta-1} T_i e^{-\sum_{i \neq s} T_i z_s^{-\theta}} e^{-T_i z_s^{-\theta}} dz_s = \\ &= \int_0^\infty \theta z_s^{-\theta-1} T_i e^{-\sum_i T_i z_s^{-\theta}} dz_s = \\ &= \frac{T_i}{\sum_i T_i} \int_0^\infty \left(\sum_{i=1..N} T_i \right) z_s^{-\theta-1} e^{-\sum_i T_i z_s^{-\theta}} dz_s \\ &= \frac{T_i}{\sum_i T_i} [e^{-z_s^{-\theta} \sum_i T_i}]_0^\infty = \frac{T_i}{\sum_i T_i} \end{aligned}$$

Property 3 follows directly from the Bayes rule.

Property 4 follows by rearranging the formula for the Frechet distribution. ■

Households

To state the household's problem formally, we need to transform this problem to guarantee integrability of the utility function and the budget constraint. To this end, given the equilibrium distribution of prices $G_n(p)$ in country n , we exploit the fact that the distribution of prices tells us the measure of goods that are available at price p . Since all variables of the model (as functions of ω) typically take identical value as long as the price is the same, without loss of generality we will index goods by their underlying prices rather than the type of good ω .

Under such reformulation, the preferences of the stand-in household from country n can be described by the following utility function:

$$U_n = \left[\int_0^\infty c_n(p)^{\frac{\sigma-1}{\sigma}} dG_n(p) \right]^{\frac{\sigma}{\sigma-1}}, \quad (1.25)$$

where σ denotes the elasticity of substitution between the goods ($\sigma > 1$), $c_n(p)$ is the price-identical consumption level of goods at price p , and $dG_n(p)$ is the measure (weight/fraction) of goods at price p .¹⁶

The problem of the household is thus to choose an integrable function $c_n(p)$ that maximizes (1.25) subject to the budget constraint:

$$\int_0^\infty p c_n(p) dG_n(p) = w_n L_n + \Pi_n, \quad (1.26)$$

where Π_n are the profits paid out by the local firms (in equilibrium Π_n will be zero),

¹⁶Note that the above formulation restricts attention to allocations in which the household chooses the same consumption of all goods that have the same price. This would be the case endogenously, but here it is build into the problem. This trick allows us to use the coarser indexation by price and guarantee integrability.

and $w_n L_n$ is compensation of labor (L_n is endowment of labor).

From the household's problem, we calculate that the ideal CPI-price index is given by:

$$P_n = \left(\int_0^\infty p^{1-\sigma} dG_n(p) \right)^{\frac{1}{1-\sigma}}. \quad (1.27)$$

A Note on Optimization with Integrals

Note that the utility maximization problem above involves a choice of an optimal function that maximizes the integral. Taking first conditions in such case may be confusing and requires some comments. For example, given the Lagrangian to the household's problem,

$$\mathcal{L} = \left[\int_0^\infty c_n(p)^{\frac{\sigma-1}{\sigma}} dG_n(p) \right]^{\frac{\sigma}{\sigma-1}} - \lambda \left(\int_0^\infty p c_n(p) dG_n(p) - \dots \right),$$

we can no longer take the pointwise derivative over $c_n(p)$ (pointwise derivative of an integral over $c_n(p)$ for any fixed p). Such derivative is zero!

So, to make any progress, we need to think about the problem more generally, and instead define the underlying variation (change) as an integrable function of the price: $dc_n(p)$. Then, the necessary condition for $c_n(p)$ to solve the utility maximization problem would clearly be that for such variation (integrable function) $dc_n(p)$, the distorted policy function $c_n(p) + \varepsilon dc_n(p)$ (where ε is some real number) achieves a local extremum at $\varepsilon = 0$. Because ε is a real number, the necessary condition for this extremum can be calculated using the standard calculus methods. This trick allows us to translate the problem to a standard one.

So, given the Lagrangian,

$$\begin{aligned} \mathcal{L}_\varepsilon = & \left[\int_0^\infty [c_n(p) + \varepsilon dc_n(p)]^{\frac{\sigma-1}{\sigma}} dG_n(p) \right]^{\frac{\sigma}{\sigma-1}} \\ & - \lambda \left(\int_0^\infty p [c_n(p) + \varepsilon dc_n(p)] dG_n(p) - \dots \right), \end{aligned}$$

we take the derivative wrt ε and evaluate it at $\varepsilon = 0$, to obtain:

$$\begin{aligned} \frac{d\mathcal{L}_\varepsilon}{d\varepsilon} \Big|_{\varepsilon=0} = & \left[\int_0^\infty c_n(p)^{\frac{\sigma-1}{\sigma}} dG_n(p) \right]^{\frac{1}{\sigma-1}} \int_0^\infty c_n(p)^{\frac{-1}{\sigma}} dc_n(p) dG_n(p) \\ & - \lambda \left(\int_0^\infty p dc_n(p) dG_n(p) \right) = 0. \end{aligned}$$

Since the function $dc_n(p)$ is any arbitrary integrable function, we observe that the above condition is equivalent to:

$$\left[\int_0^\infty c_n(p)^{\frac{\sigma-1}{\sigma}} dG_n(p) \right]^{\frac{1}{\sigma-1}} c_n(p)^{\frac{-1}{\sigma}} = p, \text{ a.s.,}$$

where ‘a.s.’ symbol means that the relationship holds almost surely (for almost all p except sets of measure zero wrt to the measure induced by G_n). This is the first order condition we are looking for.

Note that the first order condition we have derived is analogous to the first order condition we would have obtained, had we approximated the integrals by summations. In the future, we will use this observation to derive the first order condition quickly.

Exercise 12 *Given the above approach to solve the HH’s problem that involves an integral, derive the formula for the price index P_n stated above.*

Firms

In each country, the efficiency z of producing good ω is assumed to be Frechet distributed random variable (see (1.21)). Given the realization of efficiency z for a particular good type, the production function takes the form

$$y = zAl^\beta \left(\int_0^\infty q(p)^{\frac{\sigma-1}{\sigma}} dG_n(p) \right)^{\frac{(1-\beta)\sigma}{\sigma-1}}, \quad (1.28)$$

where l is labor input, $q(\cdot)$ denotes intermediate inputs, and A denotes a constant.

Competitive firms can use the above CRS technology¹⁷ to produce goods, and upon paying the iceberg transportation cost, they can ship these goods to all other countries in the world. Because production of the good is assumed to be constant returns to scale, the number of firms is undetermined. The goal of what follows is to impose conditions on the distribution of prices G_n , so that it is consistent with the described above competitive supply-side structure of the model.

Because of the constant returns to scale assumption, the marginal cost (or per unit cost) $v_{ni}(\omega)$ of producing good ω in country i for country n is here a sufficient summary of the production process. Conditional on the realization of z for a given good, the marginal cost is given by

$$v_{ni}(z) = \frac{d_{ni}c_i}{z}, \quad (1.29)$$

where

$$c_i = w_i^\beta P_i^{1-\beta} \quad (1.30)$$

is a per unit cost common across goods, and A has been chosen to soak up constants so that no constant appears in the formula $c_i = w_i^\beta P_i^{1-\beta}$.

Exercise 13 *Derive the formula for $v_{ni}(z)$ stated above from the underlying unit cost minimization problem, and calculate the value of A so that $c_i = w_i^\beta P_i^{1-\beta}$.*

Now, letting \mathcal{V}_{ni} denote the random variable describing the *marginal cost of producing good ω in country i for country n* , and letting \mathcal{Z}_i denote the underlying efficiency draw (in country i), the distribution of the random variable \mathcal{V}_{ni} can be

¹⁷All firms will be subject to the same z .

derived as follows:

$$\begin{aligned}\Pr(\mathcal{V}_{ni} \leq v_{ni}) &= \Pr\left(\frac{d_{ni}c_i}{\mathcal{Z}_i} \leq v_i\right) = \\ &= \Pr\left(\mathcal{Z}_i \geq \frac{d_{ni}c_i}{v_{ni}}\right) = 1 - F\left(\frac{d_{ni}c_i}{v_{ni}}\right).\end{aligned}\tag{1.31}$$

The lowest price \mathcal{P}_n of good ω from all possible sources is a random variable linked to \mathcal{V}_{ni} ,

$$\mathcal{P}_n = \min_{i=1..N} \{\mathcal{V}_{ni}\}.$$

Using analogous steps to the proof of Property 2, we establish that the distribution of the random variable \mathcal{P}_n is given by

$$G_n(p) = 1 - e^{-\Phi_n p^\theta},\tag{1.32}$$

where

$$\Phi_n \equiv \sum_i T(c_i d_{ni})^{-\theta}.\tag{1.33}$$

Exercise 14 *Derive the formula for $G_n(p)$ stated above.*

Finally, we analytically calculate the aggregate price index using (1.30). Plugging in the distribution function, we obtain

$$\begin{aligned}P_n^{1-\sigma} &= \int_0^\infty p^{1-\sigma} \Phi_n \theta p^{\theta-1} e^{-\Phi_n p^\theta} dp \\ &= \int_0^\infty \Phi_n p^{\theta-\sigma} e^{-\Phi_n p^\theta} dp.\end{aligned}$$

Using substitution:

$$\Phi_n p^\theta = u; \quad dp = \frac{p^{1-\theta}}{\theta \Phi_n} du; \quad p = \left(\frac{u}{\Phi_n}\right)^{\frac{1}{\theta}}$$

we obtain

$$P_n^{1-\sigma} = \int_0^\infty \left(\frac{u}{\Phi_n}\right)^{\frac{1-\sigma}{\theta}} e^{-u} du.$$

From the definition of the gamma function Γ

$$\Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t} dt$$

we get

$$\begin{aligned} P_n^{1-\sigma} &= (\Phi_n^{-\frac{1}{\theta}})^{1-\sigma} \int_0^\infty u^{\frac{1-\sigma}{\theta}} e^{-u} du \\ &= (\Phi_n^{-\frac{1}{\theta}})^{1-\sigma} \Gamma\left(\frac{\theta+1-\sigma}{\theta}\right), \end{aligned}$$

and

$$P_n = \Phi_n^{-\frac{1}{\theta}} \left[\Gamma\left(\frac{\theta+1-\sigma}{\theta}\right) \right]^{\frac{1}{1-\sigma}} = \gamma \Phi_n^{-\frac{1}{\theta}}. \quad (1.34)$$

where $\gamma = \left[\Gamma\left(\frac{\theta+1-\sigma}{\theta}\right) \right]^{\frac{1}{1-\sigma}}$. (We need to assume $\theta+1-\sigma > 0$; otherwise the above integral is not well defined.)

Exercise 15 *Show that the mean of the Frechet distribution is $T^{\frac{1}{\theta}} \Gamma(1 - \frac{1}{\theta})$.*

Market Clearing and Feasibility

The aggregate resource constraint says that demand for labor in every country equals the supply of labor in that country. This condition is difficult to state because we miss the link between the realized productivity z , type of good ω , and the realized market price for this good p .

To work around this problem, we instead note that since production function is Cobb-Douglas, fraction β of the expenditures of the entire world on home goods is equal to the total compensation of labor producing these goods (=compensation of

labor in country i), and $(1 - \beta)$ fraction is equal to payments to intermediate goods:

$$(\text{payments to labor}) \quad w_i L_i = \beta \sum_n \left(\frac{X_{ni}}{X_n} \right) X_n. \quad (1.35)$$

$$(\text{payments for intermediate goods}) = (1 - \beta) \sum_n \left(\frac{X_{ni}}{X_n} \right) X_n$$

Next, we note that by definition the total expenditures of country n , X_n , are the total final expenditures of consumers on all goods, which equal $w_i L_i$, *plus* total expenditures of home producers on all intermediate goods (home and foreign). Thus, by equation (1.35), we have:

$$\begin{aligned} X_n &= w_i L_i + (1 - \beta) \sum_n \left(\frac{X_{ni}}{X_n} \right) X_n \\ &= w_i L_i + \frac{(1 - \beta)}{\beta} w_i L_i \\ &= \frac{w_i L_i}{\beta}. \end{aligned}$$

Finally, using the above, we derive:

$$L_i = \frac{w_i L_i}{w_i} = \frac{\beta \sum_n \left(\frac{X_{ni}}{X_n} \right) X_n}{w_i} = \frac{\sum_n \left(\frac{X_{ni}}{X_n} \right) w_n L_n}{w_i}.$$

The above condition is not yet sufficient to define equilibrium because it involves an endogenous term $\frac{X_{ni}}{X_n}$ that needs to be linked to other equilibrium objects. The following lemma comes handy to fill this gap:

Lemma 16

$$\frac{X_{ni}}{X_n} = \pi_{ni} = \frac{T_i (c_i d_{ni})^{-\theta}}{\sum_i T_i (c_i d_{ni})^{-\theta}}, \quad (1.36)$$

where π_{ni} is the probability that the goods offered by country i to country n has the lowest price (and is sold to country n).

Proof. π_{ni} can be calculated analogously to the proof of Property 2. Note that

$$\pi_{ni} = \Pr(\mathcal{P}_{ni} \leq \min_{s \neq i} \mathcal{P}_{ns}).$$

To prove $\frac{X_{ni}}{X_n} = \pi_{ni}$, we need to know the distribution of the price of a good conditional on country i selling this good in country n . It turns out that this distribution is independent from the source, and is equal to the unconditional distribution G_n . The result is a consequence of Property 3, and the derivation of this fact is left as an exercise (follows the proof of Property 2). The conditional distribution of prices $\mathcal{G}_{ni}(p)$ is defined as follows:

$$\mathcal{G}_{ni}(p) \equiv \frac{\int_0^p \Pi_{s \neq i} [1 - G_{ns}(q)] dG_{ni}(q)}{\pi_{ni}} \quad (1.37)$$

This is enough to prove $\frac{X_{ni}}{X_n} = \pi_{ni}$. (Why?) ■

By the above lemma, the final labor market clearing condition is thus given by:

$$L_i = \frac{\sum_n \frac{T_i(c_i d_{ni})^{-\theta}}{\sum_i T(c_i d_{ni})^{-\theta}} w_n L_n}{w_i}. \quad (1.38)$$

Exercise 17 Prove that $\mathcal{G}_{ni}(p) = G_n(p)$ and derive π_{ni} .

Exercise 18 Assume that there is a competitive sector that produces non-tradable service goods using the following production function:

$$y = Al^\beta \left(\int_0^\infty q(p)^{\frac{\sigma-1}{\sigma}} dG_n(p) \right)^{\frac{(1-\beta)\sigma}{\sigma-1}},$$

and assume that the utility function of the household is Cobb-Douglas in tradable and non-tradable components, i.e.:

$$U_n = C_n^\alpha \left[\int_0^\infty c_n(p)^{\frac{\sigma-1}{\sigma}} dG_n(p) \right]^{\frac{(1-\alpha)\sigma}{\sigma-1}},$$

where α is the share of non-tradable goods, and C_n is consumption of the non-tradable good in country n . Furthermore, assume that labor is perfectly mobile across the two sectors producing tradable and non-tradable good. Under this modification, derive the modified labor market clearing condition. *HINT: The formula is in the Eaton and Kortum paper. You are asked to derive it.*

Equilibrium

Having laid out the economy, we next define the equilibrium.

Definition 19 *Competitive equilibrium* in this economy is:

- wages $(w_n)_{n=1..N}$ and aggregate prices $(P_n)_{n=1..N}$,

such that

- given (1.30) and (1.32), $(w_n)_{n=1..N}$ and $(P_n)_{n=1..N}$ are consistent with (1.38) and (1.34).

Given the equilibrium wage vector and aggregate price vector, all other equilibrium objects can be calculated from maximization of (1.25) subject to (1.34), and equations (1.36) and (1.37).

Computation, Existence and Uniqueness of Equilibrium

As usually, the definition of equilibrium defines a fixed point problem. The proof of uniqueness and existence thus requires to show that the fixed point exists and is unique. For details, see Alvarez and Lucas (2007).

Similarly to the Armington model, we can construct here an iterative numerical algorithm to solve for equilibrium. The sketch of the numerical algorithm would be as follows: (i) Guess wages w_n and aggregate prices P_n , (ii) Using (1.30), solve for wages from (1.38) and aggregate prices from (1.34), (iii) Iterate until convergence.¹⁸

¹⁸Use sluggish updating rule if necessary: $\text{update} = \lambda \times (\text{new value}) + (1-\lambda) \times (\text{old value})$.

Exercise 20 Write a MATLAB code that mimics the setup from exercise 6, and solves the EK model numerically.

Predictions for Trade

The following proposition summarizes the key predictions of the EK model for trade. Interestingly enough, the model turns out to have isomorphic predictions to the Armington model. The gravity equation is governed by different parameters of the model, but it has exactly the same analytical form after we relabel the parameters. Below, we study why and under which conditions this is the case.

Proposition 21 *In the Eaton-Kortum model, bilateral trade flows are governed by the following gravity equation*

$$\frac{X_{ni}}{X_n} = X_i \frac{\left(\frac{d_{ni}}{P_n}\right)^{-\theta}}{\sum_n \left(\frac{d_{ni}}{P_n}\right)^{-\theta} X_n}. \quad (1.39)$$

In particular, when iceberg transportation cost is symmetric, i.e. $d_{ni} = d_{in}$, all $i, n = 1, \dots, N$, the gravity equation is given by

$$X_{ni} = \frac{X_n X_i}{\sum_n X_n} \left(\frac{d_{ni}}{P_i P_n}\right)^{-\theta}. \quad (1.40)$$

Proof. Using equation (1.36) and equation (1.34), the balanced trade condition implies:

$$X_i = \sum_n X_{ni} = \sum_n \frac{X_{ni}}{X_n} X_n = T_i c_i^{-\theta} \sum_n \frac{d_{ni}^{-\theta} X_n}{\Phi_n} = T_i c_i^{-\theta} \sum_n \left(\frac{\gamma d_{ni}}{p_n}\right)^{-\theta} X_n,$$

and thus

$$T_i c_i^{-\theta} = \frac{X_i}{\sum_n \left(\frac{\gamma d_{ni}}{p_n}\right)^{-\theta} X_n}.$$

Combining the above with (1.36) and the definition of Φ_n (1.33), we derive:

$$\frac{X_{ni}}{X_n} = \frac{T_i(c_i d_{ni})^{-\theta}}{\sum_i T_i(c_i d_{ni})^{-\theta}} = \frac{\frac{1}{\sum_n (\frac{d_{ni}}{P_n})^{-\theta} X_n} (d_{ni})^{-\theta}}{P_n^{-\theta}} X_i = \frac{(\frac{d_{ni}}{P_n})^{-\theta}}{\sum_n (\frac{d_{ni}}{P_n})^{-\theta} X_n} X_i.$$

Similarly to the Armington model, we can derive that

$$\begin{aligned} P_n &= \gamma \left(\sum_i T(c_i d_{ni})^{-\theta} \right)^{-\frac{1}{\theta}} \\ P_n &= \left(\sum_i \frac{(\gamma d_{ni})^{-\theta}}{\sum_n (\frac{\gamma d_{ni}}{P_n})^{-\theta} X_n} X_i \right)^{-\frac{1}{\theta}} \end{aligned}$$

and so

$$P_n = \left(\sum_i \left(\frac{\gamma d_{ni}}{\Theta_i} \right)^{-\theta} X_i \right)^{-\frac{1}{\theta}}$$

where

$$\Theta_i = \left(\sum_n \left(\frac{\gamma d_{ni}}{P_n} \right)^{-\theta} X_n \right)^{-\frac{1}{\theta}}.$$

Thus, in the special case of $d_{ni} = d_{in}$, we have

$$P_n \left(\sum_n X_n \right)^{-\frac{1}{\theta}} = \Theta_n,$$

and

$$X_{ni} = \frac{X_n X_i}{\sum_n X_n} \left(\frac{d_{ni}}{P_i P_n} \right)^{-\theta}.$$

■

Discussion

Even though equation (1.39) is structurally identical to the one in the Armington model, it is not the same. In the Armington model, the critical parameter governing trade was the elasticity of substitution σ between domestic and the foreign good. In

contrast, in the EK model, σ is irrelevant, and the critical parameter is the dispersion of technologies θ . As a result, even though the model has the same implications for trade, different assumptions on the physical environment give rise to these predictions.¹⁹ Because of this peculiar isomorphism, the Armington model can be thought of as a reduced form representation of the EK model.

The reason why dispersion θ turns out critical for trade in the EK model can be understood as follows. When there is little dispersion in productivities across goods, prices presented by all alternative sources of country n are very close to the current cheapest source i , and thus country n almost immediately switches to an alternative source when the cheapest source becomes more expensive due to an increase in d_{ni} . As a result, trade flows between country n and country i fall drastically in response to d_{ni} , and this sensitivity falls as the dispersion of productivity and prices rises.

The above reasoning explains why θ is important in the EK model, but it does not explain why σ does not matter at all (does not show up in the formula). We will tackle this problem in the next paragraph by considering a two-country version of the model.

Two-Country Case

In this section, we simplify the model and assume that there are two symmetric countries, i.e. $T_1 = T_2 = 1$, $d_{12} = d_{21} = d$, $d_{11} = d_{22} = 0$, $\beta = 1$ (labor is the only production factor). The goal is to understand better the workings of the EK model. The two country case is much easier to analyze intuitively because it is possible to map the probabilistic EK formulation onto the deterministic efficiency schedules a la Dornbusch, Fisher and Samuelson (1977).

To obtain the DFS ordering, we calculate the probability that the relative produc-

¹⁹The important lesson is that the predictions of the Armington model and the EK model can be reconciled once we think of σ in the Armington model as capturing also the substitution occurring at the production level. Under such interpretation, EK model shows that such effect is important, but on the aggregate level can be directly mapped onto elasticity without much loss of generality.

tivity is smaller than some cutoff value a and assume the probability of this event to be the index value of the pivotal good — since goods are defined on a $[0,1]$ interval, this probability does define a total order²⁰ in a mathematical sense. Formally, a good index ω is defined by the following relation:

$$\omega(a) \equiv \Pr\left(\frac{Z_2}{Z_1} \leq a\right) = \Pr(Z_2 \leq aZ_1),$$

which solves to

$$\omega(a) = \frac{1}{1 + a^{-\theta}}.$$

Inverting the above expression, we obtain the mapping from the space of indices to the corresponding relative productivities:

$$a(\omega) = \left(\frac{1 - \omega}{\omega}\right)^{-\frac{1}{\theta}}.$$

Exercise 22 *Formally derive the above expression for $\omega(a)$.*

Given the $a(\omega)$ schedule, we next recover the underlying absolute productivity schedules. From symmetry, consistency with the expression for $a(\omega)$ requires that:

$$\begin{aligned} z_1(\omega) &= A\omega^{-\frac{1}{\theta}}, \\ z_2(\omega) &= A(1 - \omega)^{-\frac{1}{\theta}}, \end{aligned}$$

where A is some constant of proportionality that we can calculate explicitly²¹.

Finally, given productivity schedules $z_i(\omega)$, we normalize wages of country 1 and 2 to 1 (by numeraire assumption and symmetry), and derive the competitive price

²⁰If X is totally ordered under \leq , then the following statements hold for all a, b and c in X : (1) If $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry); (2) If $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity); (3) $a \leq b$ or $b \leq a$ (totality).

²¹The mean of the Frechet distribution is $\mu = T^{\frac{1}{\theta}} \Gamma(1 - \frac{1}{\theta}) = \Gamma(1 - \frac{1}{\theta})$. Thus, we can calculate A from $\int_{[0,1]} A\omega^{-\frac{1}{\theta}} d\omega = \mu$. Integrating, we obtain $[A\frac{\theta}{\theta-1}\omega^{\frac{\theta-1}{\theta}}]_{[0,1]} = A\frac{\theta}{\theta-1}$, and so $A = \frac{\theta-1}{\theta}\Gamma(1 - \frac{1}{\theta})$.

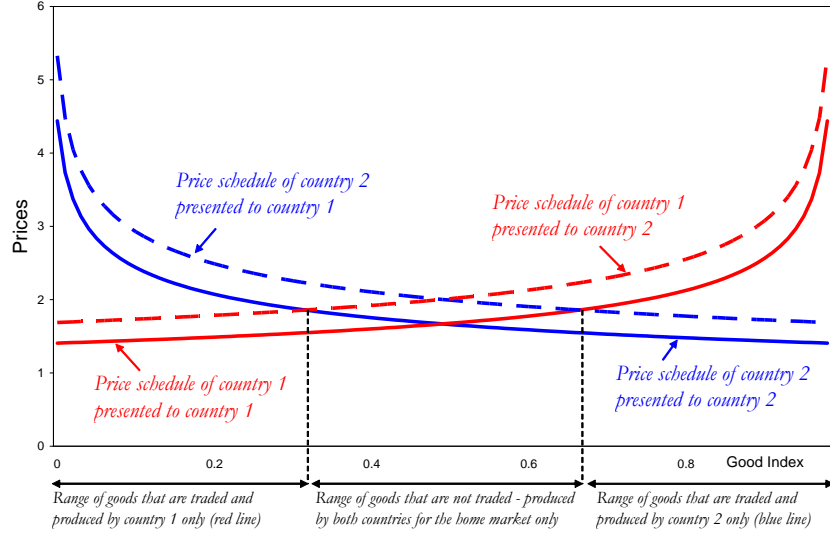


Figure 1.4: Price schedules and specialization pattern in the EK model.

schedules $p_{ni}(\omega)$ of each good ω presented by country i to country n (=marginal cost):

$$\begin{aligned}
 p_{11}(\omega) &= \frac{\omega^{\frac{1}{\theta}}}{A}, \\
 p_{21}(\omega) &= d \frac{\omega^{\frac{1}{\theta}}}{A}, \\
 p_{22}(\omega) &= \frac{(1-\omega)^{\frac{1}{\theta}}}{A}, \\
 p_{12}(\omega) &= d \frac{(1-\omega)^{\frac{1}{\theta}}}{A},
 \end{aligned}$$

where $A = \frac{\theta-1}{\theta} \Gamma(1 - \frac{1}{\theta})$.

In the EK model, the goods are always produced by the lowest cost supplier. So, it must be true that country 1 is the sole producer of all the goods for which $p_{21}(\omega) \geq p_{22}(\omega)$, and country 2 is the sole producer of goods for which $p_{12}(\omega) \geq p_{11}(\omega)$. All remaining goods are produced by both countries, and are *not* traded internationally. Figure 1.4 illustrates the obtained this way ranges of goods.

As we can see from Figure 3, in order to fully characterize the specialization pattern, we need to calculate two cutoff values. The first cutoff determines the range of goods that are produced by country 1, and the second cutoff determines the range of goods that are produced by country 2. The goods which are between these cutoffs are produced by both countries, but for the home market only. Using the formulas for prices, we can calculate these cutoff values as follows²²:

$$\begin{aligned} p_{21}(\bar{\omega}_1) &= p_{22}(\bar{\omega}_1), \\ \bar{\omega}_1 &= \frac{d^{-\theta}}{1 + d^{-\theta}} \text{ (first cutoff),} \end{aligned}$$

$$\begin{aligned} p_{12}(\bar{\omega}_2) &= p_{11}(\bar{\omega}_2), \\ \bar{\omega}_2 &= \frac{1}{1 + d^{-\theta}} \text{ (second cutoff).} \end{aligned}$$

To better understand how the EK model differs from the Armington model (Figure 4), it is instructive to derive the share of expenditures of country 1 on country 2 goods for both models. As illustrated in Figure 4, we can map the Armington model onto the DFS framework by dividing the space of goods into two equal parts. On each non-overlapping half of the ω domain both countries present a finite and identical price p at home and pd abroad, and on the other half they present an infinite price. Clearly, because the prices are constant on each interval, once we integrate over all goods, we obtain this way an Armington model with two representative goods. Let's now calculate the expenditure shares for each model.

In the Armington model, the expenditures on each good from the continuum are

²²Note that $\frac{d^{-\theta}}{(1+d^{-\theta})} < \frac{1}{(1+d^{-\theta})}$ and so the range of goods that are not traded is non-empty as long as $d > 1$. The endogenous set of goods that are actually traded is one of the key differences between the EK model and the Armington model.

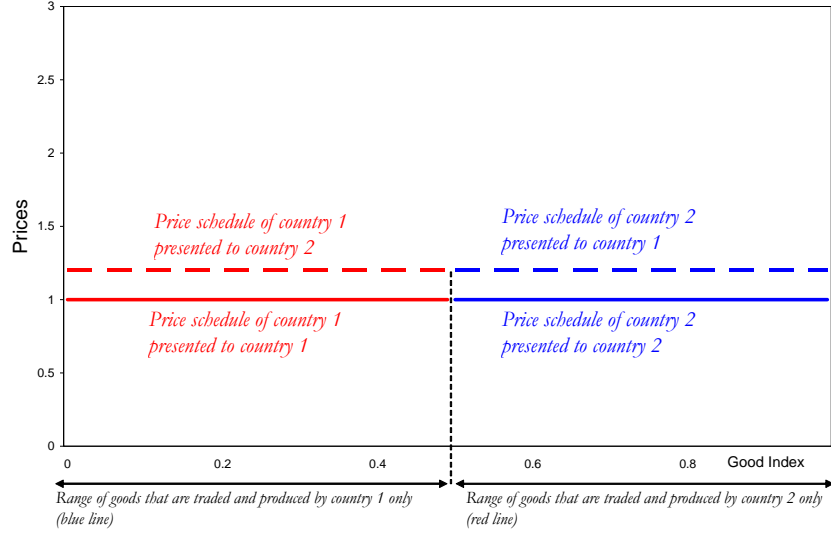


Figure 1.5: Price schedules and specialization in the Armington model.

given by

$$x(\omega) = X_1 \left(\frac{p(\omega)}{P} \right)^{1-\sigma}. \quad (1.41)$$

If we normalize the wages in both countries to 1, by symmetry we have $p_{11}(\omega) = 1$, $p_{21}(\omega) = d$, and so the price index is given by

$$\begin{aligned} P_1 &= \left(\int_0^{1/2} 1^{1-\sigma} + \int_{1/2}^1 d^{1-\sigma} \right)^{\frac{1}{1-\sigma}} = \\ &= \left(\frac{1}{2} \right)^{\frac{1}{1-\sigma}} (1 + d^{1-\sigma})^{\frac{1}{1-\sigma}}. \end{aligned}$$

Further normalizing the endowment vector (L_n) to 1, we can calculate the share of expenditures of country 1 on the goods produced by country 2 by integrating the

demand function as follows²³:

$$\begin{aligned}\frac{X_{12}}{X_1} &= \frac{1}{P^{1-\sigma}} \left(\int_{1/2}^1 d^{1-\sigma} \right) = \\ &= \frac{1}{\frac{1}{2}(1 + d^{1-\sigma})} \frac{1}{2} d^{1-\sigma} = \\ &= \frac{d^{1-\sigma}}{1 + d^{1-\sigma}}\end{aligned}$$

As we can see, in consistency with our earlier results, the parameter that determines the effect of the iceberg transportation cost on trade in the Armington model is the elasticity of substitution σ . Precisely, the more substitutable the goods are, the more the share of expenditures $\frac{X_{12}}{X_1}$ is affected by the change in d , as indicated by the derivative evaluated at $d = 1$:

$$\frac{d(\frac{X_{12}}{X_1})}{dd} = (1 - \sigma).$$

We next derive an analogous expression for the EK model. The calculations are more complicated because we have to integrate over the price schedules that aren't constant. In addition, the cutoffs determining the endogenous set of imported goods depend on d .

²³Notice that the result we derive below is consistent with the gravity equation derived for the general case (see previous section).

Normalizing wages and endowments, the price index can be computed as follows:

$$\begin{aligned}
P_1 &= \left[\int_0^{\frac{1}{1+d^{-\theta}}} p_{11}(\omega)^{1-\sigma} d\omega + \int_{\frac{1}{1+d^{-\theta}}}^1 p_{12}(\omega)^{1-\sigma} d\omega \right]^{\frac{1}{1-\sigma}} = \\
&= \left[\int_0^{\frac{1}{1+d^{-\theta}}} \left(\frac{\omega^{\frac{1}{\theta}}}{A} \right)^{1-\sigma} d\omega + \int_{\frac{1}{1+d^{-\theta}}}^1 \left(d \frac{(1-\omega)^{\frac{1}{\theta}}}{A} \right)^{1-\sigma} d\omega \right]^{\frac{1}{1-\sigma}} = \\
&= [(\ast) + (\ast\ast)]^{\frac{1}{1-\sigma}} = \\
&= \frac{(1-\sigma+\theta)^{\frac{1}{\sigma-1}}}{A\theta^{\frac{1}{\sigma-1}}} \left[\left(\frac{1}{1+d^{-\theta}} \right)^{\frac{1-\sigma+\theta}{\theta}} + \frac{d^{-\theta}}{(1+d^{-\theta})^{\frac{1-\sigma+\theta}{\theta}}} \right]^{\frac{1}{1-\sigma}} = \\
&= A^{-1} \left(\frac{\theta}{1-\sigma+\theta} \right)^{\frac{1}{1-\sigma}} (1+d^{-\theta})^{-\frac{1}{\theta}}
\end{aligned}$$

where

$$\begin{aligned}
(\ast) &= \int_0^{\frac{1}{1+d^{-\theta}}} \left(\frac{\omega^{\frac{1}{\theta}}}{A} \right)^{1-\sigma} d\omega = \\
&= A^{\sigma-1} \int_0^{\frac{1}{1+d^{-\theta}}} \omega^{\frac{1-\sigma}{\theta}} d\omega = \\
&= A^{\sigma-1} \left[\frac{\theta}{1-\sigma+\theta} \omega^{\frac{1-\sigma+\theta}{\theta}} \right]_0^{\frac{1}{1+d^{-\theta}}} = \\
&= \frac{A^{\sigma-1}\theta}{1-\sigma+\theta} \left(\frac{1}{1+d^{-\theta}} \right)^{\frac{1-\sigma+\theta}{\theta}},
\end{aligned}$$

$$\begin{aligned}
(\ast\ast) &= \int_{\frac{1}{1+d^{-\theta}}}^1 \left(d \frac{(1-\omega)^{\frac{1}{\theta}}}{A} \right)^{1-\sigma} d\omega = \\
&= A^{\sigma-1} d^{1-\sigma} \int_{\frac{1}{1+d^{-\theta}}}^1 (1-\omega)^{\frac{1-\sigma}{\theta}} d\omega = \\
&= A^{\sigma-1} d^{1-\sigma} \left[-\frac{\theta}{1-\sigma+\theta} (1-\omega)^{\frac{1-\sigma+\theta}{\theta}} \right]_{\frac{1}{1+d^{-\theta}}}^1 = \\
&= \frac{A^{\sigma-1}\theta}{1-\sigma+\theta} \frac{d^{-\theta}}{(1+d^{-\theta})^{\frac{1-\sigma+\theta}{\theta}}}.
\end{aligned}$$

Given the price index, the expenditure share is given by:

$$\frac{X_{12}}{X_1} = \frac{1}{P^{1-\sigma}} \left(\int_{\frac{1}{1+d^{-\theta}}}^1 p_{12}(\omega)^{1-\sigma} d\omega \right) = \frac{d^{-\theta}}{1+d^{-\theta}}.$$

Exercise 23 *By integrating over the price schedule, explicitly derive the expression for $\frac{X_{12}}{X_1}$ stated above. (NOTE: You are not allowed to use the gravity equation.)*

In consistency with our earlier findings, the result is that even though σ is critical for the demand for each good (as indicated by formula for $p_{ni}(\omega)$), the expression for the share of expenditures on imported goods turns out to be independent from σ .

To understand why this happens intuitively, we next shut down the extensive marginal (cutoff effect), and study the adjustment along the intensive margin (how much of each good to purchase). Because σ shows up in the expression for the demand, our conjecture is that along the intensive margin σ is critical in a similar fashion as in the Armington model, but it must be the extensive margin of changing cutoff values that offsets it. Our goal is to formally confirm this intuition.

To this end, we will start with the fixed cutoff values for the case $d = 1$ (both cutoffs at $\frac{1}{2}$), and increase d by some Δd while keeping the cutoffs unchanged at $\frac{1}{2}$. As expected, we in fact obtain an analogous expression to the Armington model:

$$\frac{X_{12}}{X_1} = \frac{\Delta d^{1-\sigma}}{1 + \Delta d^{1-\sigma}}.$$

This confirms our intuition that it is the offsetting effect of the extensive margin that makes σ drop out from the final expression, and the extensive margin transforms the gravity equation to an isomorphic form in which σ plays no role. In a sense, the extensive margin depends on σ in an exactly offsetting effect so that in the overall σ drops out. Clearly, this result hinges on the specific functional forms for productivity distribution, and is not a general feature of the DFS model. Nevertheless, it is

comforting that in some plausible class of parameterization Armington model can be thought of as the reduced form of the EK model.

Exercise 24 *Derive the above expression $\frac{X_{12}}{X_1} = \frac{\Delta d^{1-\sigma}}{1+\Delta d^{1-\sigma}}$. HINT: Do not forget that you also need to calculate the counterfactual price index P corresponding fixed cutoff values.*

Last but not least, we should mention that Eaton and Kortum estimate the parameter θ from the data on the dispersion of retail prices of commodities across countries. Their result $\theta \simeq 8$ is consistent with the high value of the elasticity parameter σ that we needed in the Armington model to account for trade patterns between Canada and US.

Endogenizing the Frechet Distribution

The parameterization of the model using the Frechet distribution holds a promise for a tractable integration of trade theory with the theory of innovation and growth. Kortum (1997) and Eaton and Kortum (1999) show how a process of innovation and diffusion can endogenously give rise to a Frechet distribution, where T_i reflects a country's stock of original (or imported ideas). The appendix at the end provides necessary probability theory background for this part.

Model of Innovation

Time is continuous and an idea is a technique to produce a certain good. The arrival of new ideas is governed by a Poisson process with time-varying intensity $aR(t)$, where we interpret $R(t)$ as the flows of R&D expenditures at time t and a is productivity of research. $R(t)$ is exogenous, but can, in principle, be endogenized²⁴.

²⁴See for example Kortum and Klette (2004) and Eaton and Kortum (2001).

A technique (=idea) is summarized by a number z , characterizing the labor requirement to produce 1 unit of output. It is assumed that upon the arrival of an idea, labor requirement Z associated with this idea is a random variable drawn from a Pareto distribution given by:

$$F(z) \equiv P(Z > z) = \begin{cases} (\frac{z}{\bar{z}})^{-\theta} & \text{if } z \geq \bar{z} \\ 1 & \text{otherwise} \end{cases}$$

where \bar{z} and θ are parameters of the distribution.

Given that the arrival process is Poisson, the number of ideas of productivity $Z > z$ discovered up to date t is distributed Poisson with parameter (see Appendix)

$$\lambda(t) = aT(t)(\frac{z}{\bar{z}})^{-\theta}, \quad z \geq \bar{z}$$

where T denotes the cumulative research effort up to time T

$$T(t) = \int_0^t aR(t)dt.$$

In the part the follows, we will find it convenient to normalize a so that $a\bar{z}^\theta = 1$, and think of $\bar{z} \rightarrow 0$. This way we can extend the domain, and have instead:

$$\lambda(t) = T(t)z^{-\theta}, \quad z \in (0, \infty).$$

Finally, we note that Pareto distribution has a convenient property that the conditional distribution on $Z \geq \hat{z}$, is also Pareto given by

$$F(z|\hat{z}) \equiv P(Z > z) = (\frac{z}{\hat{z}})^{-\theta}.$$

We will use this property in the proof below.

Distribution of the Best and the Second-Best Idea Up to Date

Given all the techniques discovered up to period t , we can rank them from the highest efficiency to the lowest, and define the underlying random variables as follows: $Z^1 \geq Z^2 \geq \dots$. Our goal is to characterize the distribution of random variables Z^1 and Z^2 in such ranking. For the EK model, it would be enough to characterize Z^1 only. However, for later use we will derive a more general characterizing the joint distribution of Z^1, Z^2 .

Proposition 25 *The joint distribution function of the best and second best technique Z^1 and Z^2 is given by*

$$G(Z_1 \leq z_1, Z_2 \leq z_2) = (1 + T(z_2^{-\theta} - z_1^{-\theta}))e^{-Tz_2^{-\theta}}, \quad (1.42)$$

where $z_1 \geq z_2$.

From the above proposition, we note that the least restrictive condition on z_2 is that $z_2 = z_1$ (as z_2 still has to be lower than z_1). Plugging in, we obtain the distribution function of the best draw that we have used in the EK model:

$$e^{-Tz_2^{-\theta}}.$$

We now turn to the proof of the above proposition.

Proof. (Courtesy of Yoichi Ueno – who nicely simplified the original proof) Suppose at period t there are n draws $(Z_n)_n$ of Z bounded by some \hat{z} , with the following order: $Z_1 \geq Z_2 \geq \dots \geq Z_n \geq \hat{z}$. In the first step, conditional on n draws and $Z_1, \dots, Z_n \geq \hat{z}$, we will calculate the conditional probability that Z_1 is above some cutoff value z_1 and Z_2 is below some cutoff value z_2 . Clearly, the conditional distribution of Z is given by $F(\zeta) = (\frac{\zeta}{\hat{z}})^{-\theta}$, $z \geq \hat{z}$. Since these events are disjoint, the probability of the event

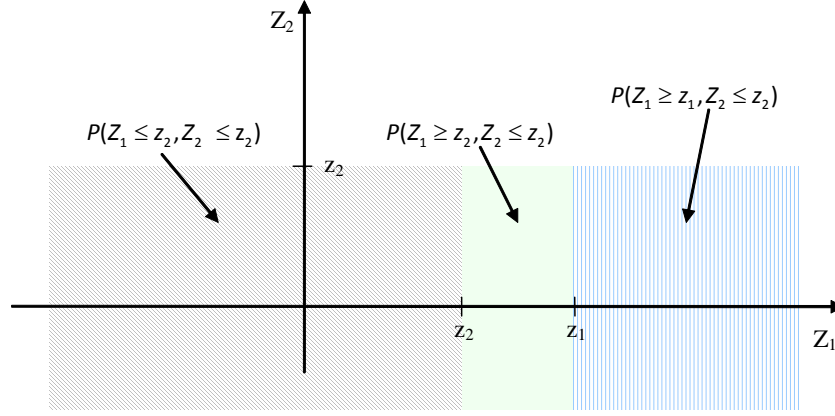


Figure 1.6: Decomposition of the distribution.

described above is given by the following Bernoulli trial formula:

$$\begin{aligned}
 P(Z_1 \geq z_1, Z_2 \leq z_2 | n, \hat{z}) &= \binom{n}{1} F(z_1) (1 - F(z_2))^{n-1} = \\
 &= n \left(\frac{z_1}{\hat{z}}\right)^{-\theta} \left(1 - \left(\frac{z_2}{\hat{z}}\right)^{-\theta}\right)^{n-1}
 \end{aligned}$$

Next, we note that since n is distributed Poisson with parameter $T\hat{z}^{-\theta}$, the unconditional distribution must be given by

$$\begin{aligned}
 P(Z_1 \geq z_1, Z_2 \leq z_2 | \hat{z}) &= \sum_{n=0}^{\infty} n \left(\frac{z_1}{\hat{z}}\right)^{-\theta} \left(1 - \left(\frac{z_2}{\hat{z}}\right)^{-\theta}\right)^{n-1} \frac{(T\hat{z}^{-\theta})^n}{n!} e^{-T\hat{z}^{-\theta}} = \\
 &= \sum_{n=1}^{\infty} \left(\frac{z_1}{\hat{z}}\right)^{-\theta} \left(1 - \left(\frac{z_2}{\hat{z}}\right)^{-\theta}\right)^{n-1} \frac{(T\hat{z}^{-\theta})^n}{(n-1)!} e^{-T\hat{z}^{-\theta}}.
 \end{aligned}$$

Collecting terms to use (1.44), we obtain:

$$\begin{aligned}
 P(Z_1 \geq z_1, Z_2 \leq z_2 | \hat{z}) &= T\hat{z}^{-\theta} \left(\frac{z_1}{\hat{z}}\right)^{-\theta} \sum_{n=1}^{\infty} \frac{(T\hat{z}^{-\theta} - Tz_2^{-\theta})^{n-1}}{(n-1)!} e^{-T\hat{z}^{-\theta}} = \\
 &= T\hat{z}^{-\theta} \left(\frac{z_1}{\hat{z}}\right)^{-\theta} e^{-Tz_2^{-\theta}} \underbrace{\sum_{n=1}^{\infty} \frac{(T\hat{z}^{-\theta} - Tz_2^{-\theta})^{n-1}}{(n-1)!} e^{-T(\hat{z}^{-\theta} - z_2^{-\theta})}}_{=1} = \\
 &= T\hat{z}^{-\theta} \left(\frac{z_1}{\hat{z}}\right)^{-\theta} e^{-Tz_2^{-\theta}} = Tz_1^{-\theta} e^{-Tz_1^{-\theta}}.
 \end{aligned}$$

Now, since the expression does not depend on \hat{z} , taking the limit $\hat{z} \rightarrow 0$, we obtain

$$P(Z_1 \geq z_1, Z_2 \leq z_2) = Tz_1^{-\theta} e^{-Tz_1^{-\theta}}.$$

Similarly, we can derive:

$$P(Z_1 \leq z_2, Z_2 \text{ anything}) = P(Z_1 \leq z_2, Z_2 \leq z_2) = e^{-Tz_2^{-\theta}}, \quad (*)$$

Since by Figure 1.6:

$$\begin{aligned}
 P(Z_1 \leq z_1, Z_2 \leq z_2) &= P(Z_1 \leq z_2, Z_2 \leq z_2) + \\
 &P(Z_1 \geq z_2, Z_2 \leq z_2) - P(Z_1 \geq z_1, Z_2 \leq z_2),
 \end{aligned}$$

we have

$$\begin{aligned}
 P(Z_1 \leq z_1, Z_2 \leq z_2) &= e^{-Tz_2^{-\theta}} + Tz_2^{-\theta} e^{-Tz_2^{-\theta}} - Tz_1^{-\theta} e^{-Tz_1^{-\theta}} = \\
 &= (1 + T(z_2^{-\theta} - z_1^{-\theta})) e^{-Tz_2^{-\theta}}.
 \end{aligned}$$

■

Exercise 26 (Optional) Derive the expression denoted by (*) in the proof above.

A Note on the Literature

This part was based on Eaton and Kortum (2002), as well as the general equilibrium analysis of this model provided by Alvarez and Lucas (2004). The material on link between the Frechet distribution and innovation comes from the paper by Bernard, Eaton, Jensen and Kortum (2003) and Eaton and Kortum textbook.

Appendix

Poisson Process

Poisson process is a jump process in the sense that a shock of a random magnitude occurs at random times with same intensity (example: arrival of phone calls or claims to a customer service office). The process $\pi(t)$ is Poisson with parameter $\lambda(t)$ (may depend on time or be constant), if it obeys the following probability conditions:

- (i) $\Pr(\text{event occurs exactly once in time interval } (t, t + dt)) = \lambda(t)dt + o(dt)$,
- (ii) $\Pr(\text{event does not occur at all in time interval } (t, t + dt)) = 1 - \lambda(t)dt + o(dt)$,
- (iii) $\Pr(\text{event does occurs more than once in time interval } (t, t + dt)) = o(dt)$,

where dt is an infinitesimal time interval and $o(dt)$ refers to terms sufficiently small relative to dt to be ignored ($\lim_{dt \rightarrow 0} \frac{o(dt)}{dt} = 0$).

Conditions (i)–(iii) imply the following distribution that the random event occurs sometime before time t is,

$$F(t) = 1 - e^{-\int_0^t \lambda(\tau) d\tau}.$$

It can be derived as follows. Let $p(t)$ denote the probability that the event does not occur up to time t . Then, p should obey the following rule (by i – iii)

$$p(t + dt) = (1 - \lambda(t)dt)p(t) + p(t)o(dt).$$

This rule says that the probability that event does not occur to time $t + dt$ is given by the probability that it does not occur up to time t , $p(t)$, and does not occur in the interval $(t, t + dt)$. Dividing both sides of the above expression by dt and taking limit $dt \rightarrow 0$, we obtain the following differential equation

$$p'(t) = -\lambda(t)p(t),$$

which solves to

$$p(t) = e^{-\int_0^t \lambda(\tau) d\tau},$$

given the initial condition $p(0) = 1$. Thus, the probability that event occurs at least once sometime up to time t is in fact given by

$$F(t) = 1 - e^{-\int_0^t \lambda(\tau) d\tau}.$$

A nice property of the Poisson process is that you can combine them together. For example, if X_n has Poisson distribution with parameter $\lambda_n(t)$, then $\sum_n^N X_n$ is also Poisson with parameter $\sum_n^N \lambda_n(t)$.

Derivation of the Poisson Distribution

Let the number of counts up to time t be denoted by $N(t)$ (which is total number of occurrence of certain "events" up to time t), e.g. arrival of phone calls to a customer service center. The distribution of the number of 'counts' up to time t is

$$P(N = n; t) = \frac{(-\int_0^t \lambda(\tau) d\tau)^n}{n!} e^{-\int_0^t \lambda(\tau) d\tau}. \quad (1.43)$$

(In particular, we note that because it is a probability distribution

$$\sum_{n=0}^{\infty} \frac{(-\int_0^t \lambda(\tau) d\tau)^n}{n!} e^{-\int_0^t \lambda(\tau) d\tau} = 1, \quad (1.44)$$

which is sometimes useful in the proofs.)

To derive the above, denote $p_n(t) = P(N = n; t)$, and note that by (i-iii) the following recursive formula holds

$$p_n(t + dt) = p_n(t)(1 - \lambda(t)dt - o(dt)) + p_{n-1}(t)(\lambda(t)dt + o(dt)).$$

Dividing by dt and letting $dt \rightarrow 0$ (using $\frac{o(dt)}{dt} = 0$), we have

$$p'_n(t) = p_n(t)(1 - \lambda(t)) + p_{n-1}(t)\lambda(t).$$

Next, we note that differential equation of the form

$$f'(x) = Af(x) + g(x), \quad (x \geq 0, g \text{ continuous})$$

solves to

$$f(t) = e^{\int_0^t A(\tau) d\tau} \left[f(0) + \int_0^t g(s) e^{-\int_0^s A(\tau) d\tau} ds \right].$$

Applying the above formula, we obtain

$$p_n(t) = e^{-\int \lambda(t) dt} \left[p_n(0) + \lambda(t) \int_0^t p_{n-1}(s) e^{-\int_0^s \lambda(\tau) d\tau} ds \right].$$

where $p_n(0) = 1$ if $n = 0$, and $p_n(0) = 0$ if $n > 0$. Using the above equation, it is easy to show by induction that (1.43) holds.

1.7 New Trade Theory

The most primitive atom responsible for exporting, production and trade is a firm. Thus, the question why countries trade naturally boils down to the question why firms decide to export. This is the starting point of the new trade theory, which takes the route of building a positive theory of industry to directly model this decision.

The above approach to model trade has been first proposed by Krugman (1980), and later developed in Helpman and Krugman (1985). Below, we first discuss the original Krugman model, as laid out in the Helpman and Krugman (1985) textbook, and then discuss the Melitz model, which merges Krugman's theory with the closed economy Hopenhayn (1992) model of industry equilibrium.

1.8 Krugman-Helpman Model

Krugman model importantly departs from the traditional trade theory. In the traditional trade theory, differences between countries give rise to trade. In contrast, in the Krugman model countries are ex-ante identical, but still decide to trade and specialize ex-post. The key driving force behind trade and ex-post specialization is the combination of increasing returns and love for variety by the consumers. In particular, the Krugman model takes a stand what a firm is: it is an ownership right to a variety, that is imperfectly substitutable with other varieties. In particular, exporting is a decision of this entity.

Model Economy

There are N countries, and each country has access to a technology of introducing a new variety (a new type of good). The space of varieties is potentially infinite, and so there is zero probability that a newly introduced variety will overlap with anything that already exists in the world (always a finite measure).

Introducing a variety takes resources modeled by a sunk cost χ . Upon introducing a variety, the entity that does so (called a firm), becomes a monopolist in producing it. We will denote the space of existing varieties in the world by $\Omega \subseteq [0, \infty)$. Note that this space can be partitioned into disjoint subsets of varieties by the source country: $\Omega = \cup_{n=1, \dots, N} \Omega_n$, where $\Omega_n \cap \Omega_m = \emptyset$, all $n, m = 1, \dots, N$.

We model trade cost by the usual iceberg transportation cost.

Households

Given the set of existing varieties, households' in country n maximize utility function given by the CES aggregator:

$$U_n = \sum_{i=1..N} \left(\int_{\Omega_i} c_{ni}(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right)^{\frac{\sigma}{\sigma-1}}, \quad (1.45)$$

subject to the budget constraint

$$\sum_{i=1..N} \int_{\Omega_i} p_{ni}(\omega) c_{ni}(\omega) = w_n L_n + \Pi_n.$$

Firms

Becoming a firm costs χ units of labor, and this cost is sunk. Upon paying it, a new variety is born and the firm is a monopolist in producing it. Given the demand function $c_{ni}(p, \omega)$ for this variety by country n households, the profit of a firm ω from country i is selling variety ω to country n is:

$$\pi_{ni}(\omega) = \max_{p_{ni}, l_i} [p_{ni} c_{ni}(p_{ni}, \omega) - w_i l_i], \quad (1.46)$$

where

$$\sum_{n=1..N} d_{ni} c_{ni}(p_{ni}, \omega) = l_i.$$

Since there is fixed cost of introducing a variety, the total profits are given by

$$\Pi_i = \pi_{ni} - \chi w_i, \text{ all } i = 1, \dots, N \quad (1.47)$$

Feasibility and Market Clearing

Market clearing condition requires that the demand for labor equals the supply of labor, and so

$$\Omega_n(\chi + l_n) = L_n, \text{ all } n = 1, \dots, N. \quad (1.48)$$

The free entry condition implies that the profits from a marginal variety are driven to zero:

$$\Pi_n = 0, \text{ all } n = 1, \dots, N. \quad (1.49)$$

Definition of Equilibrium

Definition of equilibrium is as follows.

Definition 27 *Equilibrium in this economy is:*

- prices $p_{ni}(\omega)$, w_i , and
- demand functions²⁵ $c_{ni}(p, \omega)$,
- and allocation $\Omega_n, l_n, c_{ni}(\omega)$,

such that

- given prices, demand function $c_{ni}(p, \omega)$ is derived from the household's problem given by (1.45),
- given demand functions, $p_{ni}(\omega)$ and l_i solve the firm's maximization problem given by (1.46),

²⁵It is not necessary to include demand function as part of the definition of equilibrium. We include it for the sake of clarity.

- equilibrium consumption $c_{ni}(\omega)$ is consistent with the demand function and prices, i.e. $c_{ni}(\omega) = c_{ni}(p_{ni}(\omega), \omega)$,
- zero profit condition (1.49) holds, and
- market clearing condition (1.48) is satisfied.

Characterization of Equilibrium and Predictions for Trade

The Krugman model predicts that the set of varieties introduced in equilibrium is independent from trade costs. For example, a country that adopts extreme protectionism will produce the same number of varieties as a country that opens up to trade. As a result, its predictions for gravity are identical to the Armington model. Below, we prove these two knife-edge results.

Proposition 28 Ω_i 's are independent on trade costs d_{ni} .

Proof. From the household problem, we can derive the demand for variety ω to be given by

$$x_{ni}(\omega) = \left(\frac{p_{ni}(\omega)}{P_n} \right)^{1-\sigma} X_n, \quad (1.50)$$

where $x_{ni}(\omega)$ denotes expenditures of HH from country n on variety ω coming from country i . Given this demand function, it is straightforward to show that profit maximization implies that the price will optimally be set by firms as a constant markup on the marginal cost:

$$p_{ni}(\omega) = \frac{\sigma}{\sigma - 1} w_i d_{ni}, \quad (1.51)$$

and profits are given by:

$$\pi_i(\omega) = \sum_{n=1..N} \frac{x_{ni}(\omega)}{\sigma} = \frac{1}{\sigma} \sum_{n=1..N} \left(\frac{\frac{\sigma}{\sigma-1} w_i d_{ni}}{P_n} \right)^{1-\sigma} X_n.$$

The zero profit condition says that

$$\pi_i(\omega) = \chi w_i.$$

Using the formula for profit function $\pi_i(\omega)$ stated above, we obtain

$$\pi_i(\omega) = \frac{1}{\sigma} \sum_{n=1}^N \left(\frac{\frac{\sigma}{\sigma-1} w_i d_{ni}}{P_n} \right)^{1-\sigma} X_n = \chi w_i$$

and thus

$$\sum_{n=1}^N \left(\frac{\frac{\sigma}{\sigma-1} d_{ni}}{P_n} \right)^{1-\sigma} X_n = \sigma w_i^\sigma \chi. \quad (1.52)$$

Next, using symmetry (all firms in country i are identical, and so $x_{ni} \equiv x_{ni}(\omega) = x_{ni}(\omega')$, all ω, ω'), we combine the above result with the fact that total spending of each country on all differentiated goods are given by:

$$\Omega_i \sum_{n=1}^N x_{ni} = w_i L_i. \quad (1.53)$$

In addition, using the definition of expenditures, and equation (1.51), we have

$$X_i \equiv \Omega_i \sum_{n=1}^N x_{ni} = \Omega_i \sum_{n=1}^N \left(\frac{\frac{\sigma}{\sigma-1} d_{ni} w_i}{P_n} \right)^{1-\sigma} X_n.$$

Combining the above with (1.53), we write

$$\Omega_i \sum_{n=1}^N \left(\frac{\frac{\sigma}{\sigma-1} d_{ni}}{P_n} \right)^{1-\sigma} X_n = w_i^\sigma L_i. \quad (1.54)$$

Finally, combining (1.52) with (1.54), we establish

$$\begin{aligned} \frac{w_i^\sigma L_i}{\Omega_i} &= \sigma w_i^\sigma \chi, \\ \Omega_i &= \frac{L_i}{\sigma \chi}. \end{aligned}$$

■

Proposition 29 *The gravity equation is the same as in the Armington model.*

Proof. Using the definition of aggregate expenditures

$$X_i = \sum_n \int_{\Omega_i} x_{ni}(\omega),$$

symmetry

$$x_{ni}(\omega) = x_{ni}(\omega'), \text{ all } \omega, \omega' \in \Omega_i,$$

and constant markup pricing (1.51), we obtain

$$X_i = w_i^{1-\sigma} \Omega_i \sum_{n=1}^N \left(\frac{\frac{\sigma}{\sigma-1} d_{ni}}{P_n} \right)^{1-\sigma} X_n$$

and thus

$$w_i^{1-\sigma} \Omega_i = \frac{X_i}{\sum_{n=1}^N \left(\frac{\frac{\sigma}{\sigma-1} d_{ni}}{P_n} \right)^{1-\sigma} X_n} \quad (1.55)$$

Using (1.50),

$$x_{ni}(\omega) = \left(\frac{p_{ni}(\omega)}{P_n} \right)^{1-\sigma} X_n,$$

and combining with (1.51), we obtain

$$\frac{\Omega_i x_{ni}}{X_n} = \Omega_i \left(\frac{\frac{\sigma}{\sigma-1} d_{ni} w_i}{P_n} \right)^{1-\sigma}$$

and thus

$$\frac{X_{ni}}{X_n} = \Omega_i w_i^{1-\sigma} \left(\frac{\frac{\sigma}{\sigma-1} d_{ni}}{P_n} \right)^{1-\sigma}. \quad (1.56)$$

Plugging in from (1.55) to (1.56), we obtain

$$\frac{X_{ni}}{X_n} = \frac{X_i}{\sum_{n=1}^N \left(\frac{\frac{\sigma}{\sigma-1} d_{ni}}{P_n} \right)^{1-\sigma} X_n} \left(\frac{\frac{\sigma}{\sigma-1} d_{ni}}{P_n} \right)^{1-\sigma} \quad (1.57)$$

Finally, following the same steps as in Armington model, it is easy to show that (1.57) simplifies to

$$X_{ni} = \frac{X_n X_i}{\sum_n X_n} \left(\frac{\frac{\sigma}{\sigma-1} d_{ni}}{P_n P_i} \right)^{1-\sigma}.$$

■

Exercise 30 Consider a 2-country version of the Krugman model, with one large country called North and one small country called South. Size is modeled by labor endowment, and so North has a larger endowment of labor than South. Except for a different labor endowment, the countries are identical.

In addition, assume that each country produces a homogeneous agricultural tradable good A , and consumers have Cobb-Douglas preferences between the agricultural good and the composite of all differentiated products C , i.e. preferences in country $n = N, S$ are given by:

$$U_n = A_n^\mu C_n^{1-\mu},$$

where

$$C_n = \left(\sum_{i=S,N} \int_{\Omega_i} c_{ni}(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right)^{\frac{\sigma}{\sigma-1}}.$$

The homogeneous good can be trade frictionlessly ($d = 1$), but the differentiated good is still subject to potentially positive iceberg transportation cost $d > 1$.

Derive analytically how the fraction of varieties produced by the South $\zeta = \frac{|\Omega_S|}{|\Omega_S| + |\Omega_N|}$ depends on the relative size of south $s = \frac{L_S}{L_N + L_S}$, where here $|\Omega_i|$ denotes the mass of differentiated varieties produced by i . *HINT: The existence of a homogeneous and tradable agricultural good that can be trade frictionlessly implies that that wages in both countries must be equal (despite size differences), thus by numeraire normalization $w_N = w_S = 1$. Cobb-Douglas preferences imply that households spend a constant fraction μ on the agricultural good and rest on the differentiated goods.*

1.9 Melitz Model

The Krugman model falls short in two important respects. On one hand, it introduces firms as primitive atoms of trade—which is good—but on the other hand, it is grossly at odds with the firm-level data. In the data, most firms do not export, and in the Krugman model they all do (or none of them does). In addition, despite anecdotal evidence, trade liberalizations or protectionism have no bearing on the industry structure. So, even though the model nicely illustrates a new motive for trade and models firms as the primitive atom of trade, it is not useful as a quantitative basis for applied work.

These shortcomings motivated Melitz (2003) to extend the simple setup by introducing fixed costs of entry and heterogeneity in the firm level productivity. In the Melitz model, it is no longer true that all firms export. Precisely, there is a cutoff level of productivity above which a firm decides to enter foreign markets, and below which it decides to sell only at home. This cutoff endogenously changes in response to trade costs, and thereby trade policies naturally lead to industry level reallocation. Also, unlike in the Krugman model, only some firms export.

By being consistent with basic industry and firm level facts, the Melitz model has recently become a basis for quantitative analysis in trade. An alternative to the Melitz model is a brilliant work by Bernard, Eaton, Jensen and Kortum (2003), and the unified framework by Eaton, Kortum and Kramartz (2006) (EKK hereafter). EKK model nests a specialized version of an asymmetric Melitz model due to Chaney (2008), BEJK model and the EK model. In what follows, we will talk about the original Melitz model and touch upon the BEJK (2003). I encourage you to read the EKK paper as well. (In the exercise at the end of the chapter, you will be asked to review the key firm level facts from the trade literature.)

Model Economy

Time is discrete and horizon infinite. There are $N + 1$ symmetric countries in the world. Symmetry is built into notation, and so all variables pertain to either a representative foreign country, denoted by f , or the domestic country, denoted by no subscript or d depending on the context (domestic and foreign country is identical).

Similarly as in the Krugman model, varieties are non-overlapping, and so the total mass $\Omega \in R_+$ of goods available in the domestic country is $\Omega = \Omega_d + N\Omega_f$, where Ω_d is the mass of varieties produced at home, and Ω_f is the mass of varieties produced by a representative foreign country and available at home.²⁶ The iceberg transportation cost is denoted by d .

Households

Following the same approach as in the EK model, we will index goods by prices rather than the type of the good ω . Since in the Melitz model no two varieties of goods overlap, we must multiply the underlying integrals by the measure of goods over which the distribution is calculated (probability measures normalize everything to 1).

Assuming that the price distribution of home varieties is given by P and of foreign varieties (available domestically) by P_f , in the stationary equilibrium the households' problem can be described by the following Bellman equation:

$$J = \max_{c_d(\cdot), c_f(\cdot)} \left[\left(\Omega_d \int_0^\infty c(p)^{\frac{\sigma-1}{\sigma}} dP_d + N\Omega_f \int_0^\infty c_f(p)^{\frac{\sigma-1}{\sigma}} dP_f \right)^{\frac{\sigma}{\sigma-1}} + \beta J \right], \quad (1.58)$$

subject to

$$\Omega_d \int_0^\infty pc(p)dP + N\Omega_f \int_0^\infty pc_f(p)dP_f = wL + \Pi.$$

²⁶Unlike in the Krugman model, here Ω is not a set – it is a positive real number describing the measure of varieties.

The equations should be self explanatory. Note that integrals are multiplied by the mass of goods Ω_d and $N\Omega_f$, as mentioned above.

Firms

Firms can freely enter into production by paying a sunk startup cost χ_e (denominated in labor units). All existing firms are assumed to be subject to an exogenous destruction rate δ , which will induce here an endogenous turnover of firms (process of continual entry and exit).

The timing of entry and production is as follows. Upon startup, a productivity draw ϕ is assigned to a new entrant (from distribution G), and the entrant decides whether to enter into production or not. The cost of acquiring the draw of ϕ is sunk, and costs χ_e units of labor. If the the entrant decides to enter, it then pays an a production setup cost χ to sell at home. If, in addition, the entrant wants to become an exporter, it pays another setup cost χ_x abroad—in each of the foreign countries it intends to sell to.

It is assumed that the cost of exporting χ_x is large enough, so that in equilibrium there are firms which decide to sell at home only. Such firms will be referred to as home or domestic producers. Other firms will be referred to as exporters.

After entry (i.e. after paying χ_e and obtaining ϕ), firm's problem can be described by the following Bellman equation:

$$\Pi(\phi) = \max\{\Pi_d(\phi) - \chi, \Pi_d(\phi) - \chi + N(\Pi_x(\phi) - \chi_x)\}, \quad (1.59)$$

where

$$\Pi_d(\phi) = \max_{p_d, l_d} [pc(p) - wl_d + \beta(1 - \delta)\Pi_d(\phi)], \quad (1.60)$$

subject to

$$c(p) = \phi l,$$

and

$$\Pi_x(\phi) = \max_{p_f, l_x} [p_f c_f(p_f) - w l_x + \beta(1 - \delta)\Pi_x(\phi)], \quad (1.61)$$

subject to

$$dc_f(p_f) = \phi l_x,$$

where $c_d(p)$ is a demand function for domestic variety and $c_f(p)$ for a foreign variety.

The above Bellman equation can be understood as follows. The first equation says that upon paying the entry cost χ_e , and seeing the assigned productivity draw ϕ , the value of the firm (=present discounted value of profits) Π is determined by the max of two alternatives: (i) production for the home market only: $\Pi_d(\phi) - \chi$, or (ii) production for both the home and the foreign market: $\Pi_d(\phi) - \chi + N(\Pi_x(\phi) - \chi_x)$.²⁷ It is easy to prove the following property:

Lemma 31 *Entry decision and exporting decision takes the form of a cutoff rule, in which all firms with a productivity draw above ϕ_d decide to produce at least at home, and all firms above ϕ_x decide to export. Moreover, $\phi_x > \phi_d$ iff $\chi d^{1-\sigma} < \chi_x$.*

Exercise 32 *Prove the above lemma.*

In what follows, we will directly build the above Lemma into notation.

Feasibility and Market Clearing

Aggregate feasibility requires that total demand for labor used in production (by all operating firms), setup cost, and startup of firms is equal to the total supply of

²⁷Note that we built into the problem the fact that the fixed costs are such that the firm never finds it optimal to export and not to sell at home. We will later make this assumption explicit by imposing a condition on the fixed cost χ_d and χ_f .

labor:

$$\begin{aligned}
L = & \chi_e \frac{\delta\Omega_d}{1 - G(\phi_d)} (\text{startups}) + \\
& + \chi \delta\Omega_d (\text{production setup at home}) + \\
& + N \chi_x \delta\Omega_d \frac{1 - G(\phi_x)}{1 - G(\phi_d)} (\text{production setup abroad}) + \\
& + \frac{\Omega_d}{1 - G(\phi_d)} \int_{\phi_d} l_d(\phi) dG(\phi) (\text{labor demand by domestic producers}) + \\
& + \frac{\Omega_f}{1 - G(\phi_x)} \int_{\phi_x} (l_d(\phi) + N l_x(\phi)) dG(\phi) (\text{labor demand by exporters}).
\end{aligned} \tag{1.62}$$

To write down this condition formally, we have used the fact that the mass of varieties is equal to the number of firms. So, Ω_d is the total mass (measure) of firms that produce both at home and export, and Ω_f is the mass of firms that also export (which by symmetry applies to the domestic country and to any foreign country).

Since in the stationary equilibrium (steady state), $\delta\Omega_d$ of firms must be replaced every period by new entrants, there must be $\frac{\delta\Omega_d}{1 - G(\phi_d)}$ startups (draws of ϕ)—as only fraction $1 - G(\phi_d)$ of startups eventually decides to produce.

Moreover, given fraction $\delta\Omega_d$ of new entrants (firms that stay and produce), the fraction of new entrants that also decides to export something is given by the conditional probability that ϕ falls above the exporting cutoff ϕ_x , conditioned on ϕ being already above the entry cutoff ϕ_d . This probability is equal to $\frac{1 - G(\phi_x)}{1 - G(\phi_d)}$, and thus the expression $\delta\Omega_d \frac{1 - G(\phi_x)}{1 - G(\phi_d)}$. Since all firms which find it profitable to export will export to all markets or not at all, we must multiply this term by N .

The last two terms are the variable demand for labor by domestic producers (firms that sell at home only) and exporters (firms that sell both at home and abroad).

The definitions of the cutoff values ϕ_d, ϕ_x imply that the firm at the cutoff level must be indifferent between the two decisions, i.e.

$$\Pi(\phi_d) = 0, \tag{1.63}$$

and

$$\Pi_d(\phi_x) - \chi w = \Pi_d(\phi_x) - \chi w + N(\Pi_x(\phi_x) - \chi_x w). \quad (1.64)$$

What's more, under symmetry the process of entry and exit implies

$$\Omega_f = \frac{1 - G(\phi_x)}{1 - G(\phi_d)} \Omega_d, \quad (1.65)$$

and the free entry and exit condition implies

$$\Pi \equiv E\Pi(\phi) - \chi_e w = 0. \quad (1.66)$$

Finally, the price distribution P can be linked to the other equilibrium by the optimal policy function of firms: $p(\phi)$ and $p_x(\phi)$. Given these functions, P (and P_f) is derived as follows:

$$P(p) = \Pr(p \leq p) = \Pr(p(\phi) \leq p) = \Pr(\phi \leq p^{-1}(p)) = G(p^{-1}(p)), \quad (1.67)$$

and similarly

$$P_f(p) = \Pr(p_f \leq p) = \Pr(p_x(\phi) \leq p) = \Pr(\phi \leq p_x^{-1}(p)) = G(p_x^{-1}(p)). \quad (1.68)$$

Equilibrium

Definition 33 *Symmetric stationary equilibrium in this economy is:*

- value functions $J, \Pi(\phi), \Pi_d(\phi), \Pi_x(\phi)$,
- policy functions $c(p), c_f(p), p(\phi), p_f(\phi), l_d(\phi), l_x(\phi)$
- distribution functions $P(p), P_f(p)$,

- measures Ω_d, Ω_f ,
- entry cutoffs ϕ_d, ϕ_x ,
- aggregate profits Π ,
- and wage w

such that

- value function J and policy functions $c(p), c_f(p)$ are derived from (1.58),
- value functions $\Pi(\phi), \Pi_d(\phi), \Pi_x(\phi)$ and policy functions $p(\phi), p_f(\phi), l_d(\phi), l_x(\phi)$ are derived from (1.59), (1.60) and (1.61),
- $P_d(p), P_f(p)$ are consistent with (1.67) and (1.68)
- $p(\phi), p_f(\phi)$ are invertible functions,
- cutoff values ϕ_d, ϕ_x , and measures Ω_d, Ω_f are consistent with (1.63), (1.64) and (1.65),
- zero profit condition (1.66) is satisfied,
- and market clearing condition (1.62) holds.

Proposition 34 *The above symmetric equilibrium exists and is unique.*

Proof. The strategy is to show that the zero profit conditions can be used to solve for the cutoffs ϕ_d and ϕ_x , and once we find the cutoffs, the remaining equilibrium objects follow. Before we proceed, we note the following. Since this a monopolistic competition model, the formula for prices is given by:

$$\begin{aligned} p(\phi) &= \frac{\sigma}{\sigma - 1} \frac{1}{\phi}, \\ p_x(\phi) &= \frac{\sigma}{\sigma - 1} \frac{d}{\phi}, \end{aligned}$$

and expenditures on individual goods by:

$$\begin{aligned} x(\phi) &= X\left(\frac{p(\phi)}{P}\right)^{1-\sigma} = X\left(\frac{\sigma-1}{\sigma}P\phi\right)^{\sigma-1}, \\ x_x(\phi) &= X\left(\frac{p_f(\phi)}{P}\right)^{1-\sigma} = d^{1-\sigma}X\left(\frac{\sigma-1}{\sigma}P\phi\right)^{\sigma-1} = d^{1-\sigma}x(\phi). \end{aligned} \quad (1.69)$$

(Note that, in fact, prices are an invertible functions of ϕ as required by the definition of equilibrium.) In particular, from (1.69), we obtain

$$\begin{aligned} \frac{x(\phi)}{x_x(\phi')} &= d^{\sigma-1}\left(\frac{\phi}{\phi'}\right)^{\sigma-1}, \\ \frac{x(\phi)}{x(\phi')} &= \left(\frac{\phi}{\phi'}\right)^{\sigma-1}, \\ \frac{x_x(\phi)}{x_x(\phi')} &= \left(\frac{\phi}{\phi'}\right)^{\sigma-1}, \end{aligned} \quad (1.70)$$

for any ϕ, ϕ' .

In the first step, we decompose the expected profits to profits earned at home and abroad:

$$E[\pi(\phi)] = (1 - G(\phi_d))E_{\phi > \phi_d}[\pi_d(\phi)] + N(1 - G(\phi_x))E_{\phi > \phi_x}[\pi_x(\phi)], \quad (1.71)$$

and calculate the conditional profits as follows:

$$\begin{aligned} E_{\phi > \phi_d}[\pi_d(\phi)] &= \frac{E_{\phi > \phi_d}[x(\phi)]}{\sigma} - \chi, \\ E_{\phi > \phi_x}[\pi_x(\phi)] &= \frac{E_{\phi > \phi_x}[x_x(\phi)]}{\sigma} - \chi_x. \end{aligned} \quad (1.72)$$

Given (1.70), and using the above expressions, we obtain

$$\begin{aligned} E_{\phi > \phi_d}[x(\phi)] &\equiv x(\phi_d)E_{\phi > \phi_d}\left[\frac{x(\phi)}{x(\phi_d)}\right] \equiv \frac{x(\phi_d)}{\phi_d^{\sigma-1}} \int_{\phi_d}^{\infty} \frac{\phi^{\sigma-1}g(\phi)}{1 - G(\phi_d)} d\phi, \\ E_{\phi > \phi_x}[x_x(\phi)] &\equiv x_x(\phi_x)E_{\phi > \phi_x}\left[\frac{x_x(\phi)}{x_x(\phi_x)}\right] \equiv \frac{x_x(\phi_x)}{\phi_x^{\sigma-1}} \int_{\phi_x}^{\infty} \frac{\phi^{\sigma-1}g(\phi)}{1 - G(\phi_x)} d\phi, \end{aligned} \quad (1.73)$$

where $g(\cdot)$ denotes the density function corresponding to $G(\cdot)$. Furthermore, assuming $\phi_d < \phi_x$ (which holds iff $d^{\sigma-1}\chi_x > \chi$), we use (1.72) and the definition of the cutoff productivity ϕ_d , to derive

$$\pi(\phi_d) = \pi_d(\phi_d) = \frac{x(\phi_d)}{\sigma} - \chi = 0,$$

and

$$x(\phi_d) = \sigma\chi. \quad (1.74)$$

Substituting from (1.73) and (1.74) into (1.72), we have

$$\begin{aligned} E_{\phi > \phi^*}[\pi_d(\phi)] &= \frac{x(\phi_d)}{\sigma(\phi_d)^{\sigma-1}} \int_{\phi_d}^{\infty} \frac{\phi^{\sigma-1}g(\phi)}{1 - G(\phi_d)} d\phi - \chi = \\ &= \chi \left[\left(\frac{\tilde{\phi}(\phi_d)}{\phi_d} \right)^{\sigma-1} - 1 \right], \end{aligned}$$

where

$$\tilde{\phi}(\phi^*) \equiv \left[\int_{\phi^*}^{\infty} \frac{\phi^{\sigma-1}g(\phi)}{1 - G(\phi^*)} d\phi \right]^{\frac{1}{\sigma-1}}.$$

The analogous expression for exporters is

$$E_{\phi > \phi_x}[\pi_x(\phi)] = \chi_x \left[\left(\frac{\tilde{\phi}(\phi_x^*)}{\phi_x^*} \right)^{\sigma-1} - 1 \right].$$

Next, we express the ex-ante zero profit condition in terms of the cutoff values by substituting out the above equations for conditional profits into (1.71) :

$$\begin{aligned} E[\pi(\phi)] &= \chi(1 - G(\phi_d)) \left[\left(\frac{\tilde{\phi}(\phi_d)}{\phi_d} \right)^{\sigma-1} - 1 \right] + \\ &+ N\chi_x(1 - G(\phi_x)) \left[\left(\frac{\tilde{\phi}(\phi_x)}{\phi_x} \right)^{\sigma-1} - 1 \right] = \chi_e. \end{aligned} \quad (1.75)$$

Furthermore, we express the exporting cutoff ϕ_x in terms of the entry cutoff. Using

the fact that

$$\begin{aligned}\pi_d(\phi_d) &= \frac{x(\phi_d)}{\sigma} - \chi = 0, \\ \pi_x(\phi_x) &= \frac{x(\phi_x)}{\sigma} - \chi_x = 0,\end{aligned}\tag{1.76}$$

we derive

$$\begin{aligned}\frac{x_x(\phi_x)}{x(\phi_d)} &= d^{1-\sigma} \left(\frac{\phi_x}{\phi_d} \right)^{\sigma-1} = \frac{\chi_x}{\chi}, \\ \phi_x &= \phi_d \left(\frac{\chi_x}{\chi} \right)^{\frac{1}{\sigma-1}} d.\end{aligned}\tag{1.77}$$

Since, ϕ_x is a linear function of ϕ_d , with a slope coefficient $\zeta = \left(\frac{\chi_x}{\chi} \right)^{\frac{1}{\sigma-1}} d$, we reduce the equilibrium fixed point problem to 1 equation in 1 unknown:

$$\chi j(\phi_d) + N \chi_x j(\zeta \phi_d) = \chi_e,\tag{1.78}$$

where

$$j(\phi_d) \equiv (1 - G(\phi_d)) \left[\left(\frac{\tilde{\phi}(\phi_d)}{\phi_d} \right)^{\sigma-1} - 1 \right].\tag{1.79}$$

To goal is to show that there is a solution (1.78), and that it is unique. Given continuity of $j(\cdot)$, it is sufficient to prove that $j(\cdot)$ strictly decreases from ∞ to 0 for $\phi_d \in (0, \infty)$. To this end, we note the following properties of $j(\cdot)$: (i) $j(\phi_d) > 0$ on $\phi_d \in (0, \infty)$, (ii) $\phi_d \rightarrow 0$, $j(\phi_d) \rightarrow +\infty$ (follows because $\tilde{\phi}(\phi_d) \rightarrow \infty$), and (iii) $\frac{j'(\phi_d)\phi_d}{j(\phi_d)} < -(\sigma - 1)$. The only nontrivial property is (iii). We show it as follows. Consider $f(\phi_d) \equiv \log j(\phi_d)$ (this is possible because $j(\phi_d)$ is positive valued function). Note that by monotonicity of \log , $(\phi_d \rightarrow \infty, j(\phi_d) \rightarrow 0)$ iff $(\phi_d \rightarrow \infty, f(\phi_d) \rightarrow -\infty)$. Since from (ii) we know $f'(\phi_d) < -\frac{(\sigma-1)}{\phi_d}$, we use the Fundamental Theorem of

Calculus to link this fact to the function itself:

$$\begin{aligned} f(b) - f(a) &= \int_a^b f'(\phi_d) < \int_a^b -\frac{(\sigma-1)}{\phi_d} = \left[-\frac{1}{2}(\sigma-1) \log \phi_d\right]_a^b = \\ &= \frac{1}{2}(\sigma-1)(\log a - \log b). \end{aligned} \quad (1.80)$$

Taking $b \rightarrow \infty$, we have $f(b) \rightarrow -\infty$.

The remaining part of the proof is straightforward and is omitted. It requires to show that given the cutoffs, we can unambiguously determine all other equilibrium objects. ■

Comparative Statics Results

The central result of the Melitz paper is that trade liberalizations lead to the industry level reallocations. Intuitively, in the Melitz model, when d goes up, consumers shift their spending onto new goods (in equation (1.69) P falls). This lowers their spending on the goods that they were purchasing so far, and unless the firm is exporting, it necessarily faces a decline in spending and profits (by equation (1.76)). Consequently, the cutoff ϕ_d goes up. Moreover, since entry cutoff goes up, entry becomes more costly. As indicated by equation (1.66), the only way a firm can break even in expectation is that profits at the top of the distribution compensate for the losses at the bottom. This is the key result that Melitz proves and illustrates in Figure 2 (reproduced below).

We next frame this result into a formal proposition.

Proposition 35 *In the equilibrium of the Melitz model, $\frac{d\phi_d}{dd} < 0$, $\frac{d\phi_x}{dd} > 0$, and there exists $\bar{\phi}$ such that for all $\phi > \bar{\phi}$, $\frac{d\pi}{dd} < 0$.*

Exercise 36 *Prove the above proposition. HINT: For the first part consider study the expression (1.78) and (1.77) using Implicit Function Theorem. For the second*

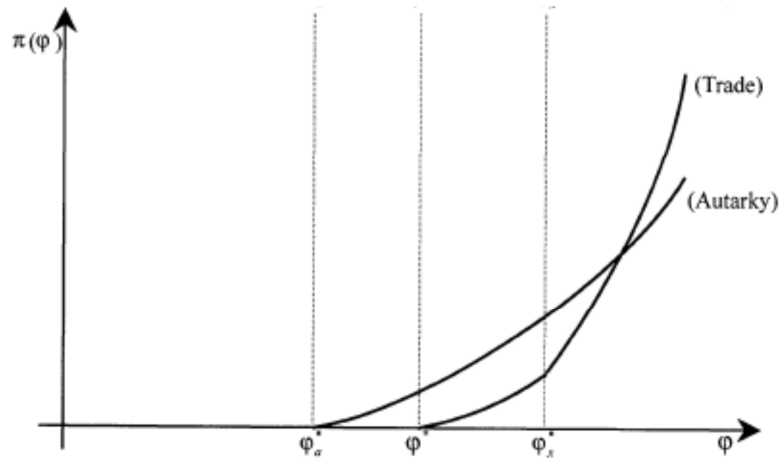


Figure 1.7: Industry level relocations in the Melitz model.

part, show

$$x_d(\phi) + Nx_x(\phi) = (1 + Nd^{1-\sigma})x_d(\phi)$$

using (1.70), and show

$$x_d(\phi) = \left(\frac{\phi}{\phi_d}\right)^{\sigma-1} \sigma \chi, \text{ all } \phi \geq \phi_d,$$

using similar arguments as in (1.76). Next, consider the change in profits of a firm that exports before and after reduction of d by some Δd . Use the fact that

$$\pi(\phi) = \frac{x(\phi)}{\sigma} - \chi - Nx_x,$$

where $x(\phi) = x_d(\phi) + Nx_x(\phi)$.

A Note on the Literature

Melitz model is based on the model of industry equilibrium by Hopenhayn (1992). The original Melitz setup has been further extended to a multi-country asymmetric framework by Chaney (2006). Unlike the Melitz model, Chaney's model is in partial equilibrium. The main contribution of his paper is to derive analytically a gravity equation. Finally, Eaton, Kortum and Kramartz (2006) propose a framework that flexibly nests all 3 models: EK model, BEJK model and Chaney's model.

1.10 Bernard, Eaton, Jensen and Kortum Model

A competing model with monopolistic producers is the model by Bernard, Eaton, Jensen and Kortum (2003). The paper extends the original framework originally due to Eaton and Kortum (2002), and introduces imperfect competition. The key idea is that a producer has an exclusive right to the technology draw he gets (unlike in the EK model), and the distribution of efficiency level in producing every good is given by the formula we have derived in the previous section from a model of endogenous innovation:

$$P(Z_1 \leq z_1, Z_2 \leq z_2) = (1 + T(z_2^{-\theta} - z_1^{-\theta}))e^{-Tz_2^{-\theta}},$$

where Z_1 is the r.v. denoting the draw of the most efficient producer and Z_2 is the r.v. denoting the draw of the second most efficient producer. The producers are Bertrand competitors, so the highest efficiency (lowest cost) producer sets the price as high as possible, but low enough so that it precludes second lowest cost producer from entry. This, of course, leads to endogenous markups, which crucially depend on the distance in efficiency between the first and second lowest cost producers. We should note that in this model innovation is confined to factor saving innovation, and does not involve product innovation as the Melitz model.

Interestingly enough, the model is set up in such a way that all simplifications

that were valid in the EK model go through in this extended framework. But, unlike using the EK model, here we can start talking about firm-level facts in addition to the aggregate facts that EK model has predictions on.

BEJK document in their paper key characteristics of an ‘exporting plant’ (*relative to* ‘non-exporting plant’) based on the data from 1992 US Census of Manufactures (200,000 plants in the sample). The key exporter facts they focus on are:

1. Very few plants report exporting anything – about 21%
2. Those that do, still sell mostly at home – 2/3 of plants in the sample export less than 10% of their total output
3. Exporting firms are on average larger (ship 5.6 more output), and appear to be about 9% more ‘productive’ (or more accurately profitable)²⁸

They show that their model can account for all the above facts qualitatively, and in many respects comes close at accounting for them quantitatively. The amazing thing is that it does so without any fixed costs of exporting. Below, you will find a short overview of the key features of the theory:

Market Structure and Ownership

- Bertrand competition between the producers within same variety (each producer owns one technology)
- Similarly to EK’02, each market is captured by the low cost supplier, but unlike in EK’02 the markup can be positive:
 - the lowest cost supplier to country n is constrained not to charge more than the second-lowest cost supplier potentially entering from any country

²⁸*Productivity* is defined here as the ratio of total value added of the plant relative to the total payroll bill of production workers (after controlling for capital/skill intensity of a given plant).

in the world

$$c_{2n}(j) = \min \left\{ c_{2nl}(j), \min_{i \neq l} \{ c_{1ni}(j) \} \right\}$$

where $c_{2n}(j)$ is the second lowest cost supplier of commodity j to country n , and l is the country of origin of the lowest-cost supplier ($= \min\{\text{second lowest cost supplier from the same country as the lowest cost supplier, min of the first lowest cost supplier to country } n \text{ from all other countries}\}$).

Prices

- The price of good j in country n is given by

$$P_n(j) = \min \{ c_{2n}(j), \frac{\sigma}{\sigma - 1} c_{1n}(j) \}.$$

- The markup is the maximal feasible markup as long as it is not higher than the MC optimal markup

Probabilistic Formulation of Technology

- To cover all possibilities, need to know the highest and second-highest efficiency draw $z_{1i}(j), z_{2i}(j)$ in each country
- Similarly to EK model, the efficiency levels are realizations of a random variable drawn from a carefully chosen distribution

$$F_i(z_1, z_2) = \Pr [Z_{1i} \leq z_1, Z_{2i} \leq z_2] = [1 + T_i(z_2^{-\theta} - z_1^{-\theta})] e^{-T_i z_2^{-\theta}},$$

for $0 \leq z_2 \leq z_1$, drawn independently across countries i and goods j (see derivation of this function from a process of endogenous innovation in the previous sections)

Cost Functions

- The cost is a realization of the following two random variables:

$$\begin{aligned} c_{1ni}(j) &= \frac{w_i}{Z_{1i}(j)} d_{ni} \\ c_{2ni}(j) &= \frac{w_i}{Z_{2i}(j)} d_{ni} \end{aligned}$$

- Given distribution of efficiency can obtain distribution of first and second lowest cost as follows:

$$\begin{aligned} G_{ni}^c(C_1 > c_1, C_2 > c_2) &= \Pr[C_1 > c_1, C_2 > c_2] \\ &= \Pr\left[Z_1 \leq \frac{w_i d_{ni}}{c_1}, Z_2 \leq \frac{w_i d_{ni}}{c_2}\right] \\ &= G\left(\frac{w_i d_{ni}}{c_1}, \frac{w_i d_{ni}}{c_2}\right). \end{aligned}$$

which solves to

$$G_{ni}^c(c_1, c_2) = [1 + T_i[w_i d_{ni}]^{-\theta}(c_2^\theta - c_1^\theta)] e^{-T_i[w_i d_{ni}]^{-\theta} c_2^\theta}$$

- The complementary distribution of the lowest and second-lowest cost regardless the source (the probability that the lowest and second-lowest cost in all countries is above c_2 + probability that in one of the countries the lowest cost is between c_1 and c_2 , second-lowest is above c_2 and in all other countries both lowest and second-lowest is above c_2) is

$$\begin{aligned} G_n^c(c_1, c_2) &= \prod_{i=1}^N G_{ni}^c(c_2, c_2) + \\ &\quad + \sum_{i=1}^N [G_{ni}^c(c_1, c_2) - G_{ni}^c(c_2, c_2)] \prod_{k \neq i} G_{nk}^c(c_2, c_2) \\ &= [1 + \Phi_n(c_2^\theta - c_1^\theta)] e^{-\Phi_n c_2^\theta} \end{aligned}$$

where

$$\Phi_n = \sum_i T_i [w_i d_{ni}]^{-\theta}.$$

- The cost distribution is

$$G_n(c_1, c_2) = 1 - G_n^c(0, c_2) - G_n^c(c_1, 0) + G_n^c(c_1, c_2)$$

- The distribution of the lowest cost regardless second lowest cost can be obtained by taking the limit $c_2 \rightarrow \infty$ of the expression above

Key Aggregate Results

1. The probability π_{ni} that country i is the lowest cost supplier to n is the same as in EK:

$$\pi_{ni} = \int_0^\infty \Pi_{k \neq i} [1 - G_{1nk}(c)] dG_{1ni}(c) = \frac{T_i (w_i d_{ni})^{-\theta}}{\Phi_n}.$$

2. The joint distribution of the lowest and second lowest cost of supplying country n , conditional on country i being the low cost supplier is independent on the source as in EK

$$G_n^c(c_1, c_2 | i) = G_n^c(c_1, c_2) = [1 + \Phi_n (c_2^\theta - c_1^\theta)] e^{-\Phi_n c_2^\theta}.$$

Unlike in the EK model, this result does not translate yet to gravity equation because there are markups. Need to know that distribution of markups is independent on the source.

3. The markup $M_n(j) = P_n(j)/C_{1n}(j)$ is the realization of a random variable M_n . Conditional on $C_2 = c_2$, the markup is Pareto distributed and independent on

c_2 . Thus, the distribution of markups conditioned on being the source is also Pareto, and in particular, independent on the source.

4. Given the above, the share that country n spends on goods from country i is π_{ni} , and gravity equation is analogous to the EK model

Firm-Level Results

1. A plant with higher efficiency is likely to have a higher markup

$$H_n(m|z) = 1 - \exp(-\Phi_n w_n^\theta z^{-\theta} (m^\theta - 1)), 1 \leq m \leq \bar{m},$$

(a plant unusually efficient relative to other producing plants tend to be unusually efficient relative to its latent competitors as well, so charges a higher markup)

2. Greater efficiency makes the producer more likely to export and to be big, explaining the correlations between size and export status that we see in the data.

Exercise 37 Read data sections of the following papers: (i) Das, Sanghamitra & Mark J. Roberts & James R. Tybout, 2001. “Market Entry Costs, Producer Heterogeneity, and Export Dynamics,” NBER Working Papers 8629, National Bureau of Economic Research, (ii) Eaton, Jonathan & Samuel Kortum & Francis Kramarz, 2004. “Dissecting Trade: Firms, Industries, and Export Destinations,” NBER Working Papers 10344, National Bureau of Economic Research, (iii) Bernard, Andrew B. & Bradford J. Jensen, 1999. “Exceptional Exporter Performance: Cause, Effect or Both?”, *Journal of International Economics*, February 1999, 47(1), pp. 1-25, and (iv) Ruhl, Kim J. & Willis Jonathan, 2007. “Convexities, Nonconvexities, and Firm Export Behavior”. The last paper you will find under the following link:

<http://editorialexpress.com/conference/MWM2008/program/MWM2008.html> (see session 20 in the conference schedule). Summarize and briefly describe the key producer-level facts about exporting and trade that emerge from this literature.

Chapter 2

International Business Cycle

“In modern developed economies, goods and assets are traded across national borders, with the result that events in one country generally have economic repercussions in others. International business cycle research focuses on the economic connections among countries and on the impact these connections have on the transmission of aggregate fluctuations. In academic studies this focus is expressed in terms of the volatility and comovement of international time series data. Examples include the volatility of fluctuation in the balance of trade, and correlation of the trade balance with output, the correlation of output and consumption across countries, and the volatility of prices of foreign and domestic goods.”, Backus, Kehoe and Kydland, 1995.

2.1 Introduction

In this chapter, we extend the time horizon of our analysis to short-run and medium-run. What occurs at these frequencies, and our long-run analysis has so far abstracted from, are business cycle fluctuations. This phenomenon brings in an additional set of facts about the comovement and volatility of aggregate quantities and prices, and introduces a new motive of international borrowing and lending to share business cycle risk and reallocate production over the business cycle.

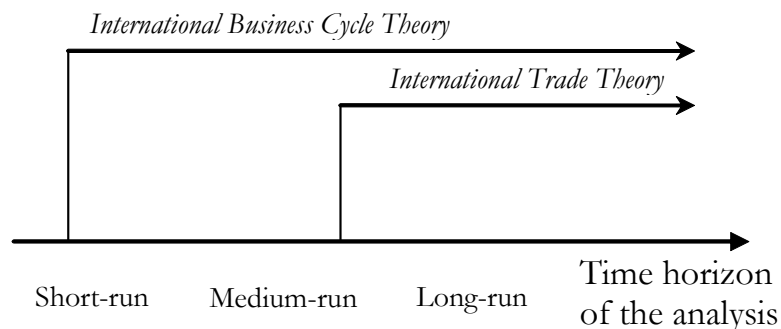


Figure 2.1: Time-horizon of the business cycle analysis.

The research in the international business cycle literature has been focused on a development of a workhorse model (laboratory) that is useful for broad-based policy analysis. Since this objective coincides with the agenda of the closed economy business cycle modeling, its open economy counterpart is often referred to as open economy macro. The central questions of the open economy macro are largely, but not only, centered around the extent to which predictions of the successful closed economy macro models hold in the open economy environments, and the extent to which these models are consistent with the additional evidence such analysis brings in (e.g. international comovement, international prices, current account dynamics).

The international business cycle literature is organized into two major branches: (i) International Real Business Cycle, and (ii) New Open Economy Models. These two classes of models are not disjoint. In fact, NOEM builds on the IRBC by adding sticky prices to the RBC model, and monetary shocks (liquidity shocks). Since in the RBC models money is neutral, without further modifications of the theory, such

shocks have no bite¹.

In this course, we will talk about IRBC and occasionally touch upon NOEM (to be discussed in Econ 872). Both types of models are based on shocks backed out from the data in a disciplined manner. In the IRBC theory, the shocks are modeled as a stochastic shift of the production function, and are backed out from the cyclical properties of the Solow residual in the data. The NOEM models add monetary shocks (liquidity shocks) that are backed out from the data on interest rates or money supply.

The shocks in the RBC models are often referred to as technology shocks. However, one should not interpret them literally, but rather think of them as capturing all sorts of unmodeled distortions that boil down to a shift of a production function in a reduced form model. For a theoretical foundation of such view, see the paper by Chari, Kehoe and McGrattan, “*Business Cycle Accounting*”, *Econometrica* 2007, Vol 75(3).²

A standard reference to RBC theory is the textbook (collection of RBC papers): “Frontiers of Business Cycle Research” by Thomas Cooley (editor)—which is a bit out of date at this point.³ As an introduction to RBC, I encourage you to read the Nobel lecture by Prescott: “Nobel Lecture, The Transformation (...)”, *JPE*, 2006, Vol114(2).

¹There are RBC models with real financial frictions that involve monetary shocks and money is not neutral. For example, see Atkeson and Kehoe (2000), “Money, Interest Rates, and Exchange Rates With Endogenously Segmented Asset Markets”, WP 605, Minneapolis Fed.

²Under such view RBC theory no longer implies that there is no space for broadly defined government intervention to stabilize business cycle. Such conclusion is only true if we literally interpret technology shocks as coming from some *exogenous* shifts in the production function (like oil shocks or hurricanes). In addition, such interpretation implies that much less can be said about the source of business cycle fluctuations. Nevertheless, the theory is still useful in providing a coherent and disciplined framework to study the effects of the business cycle on various aspect of economic activity.

³Especially in terms of solution methods.

2.2 Business Cycle Measurement

Here, we develop measurement tools that allow us to summarize the facts about business cycle fluctuation. Specifically, we develop methods that isolate the long-run properties (low-frequency) of the time-series from the short-run properties (high frequency).

The business cycle literature typically defines the business cycle as a phenomenon occurring at the frequencies 2 quarters to 8 years. To narrow down the focus to this domain, we must find an appropriate detrending method that will allow us to focus attention on this particular frequency domain.

The standard technique in the literature has been to use an HP filter. HP filter can flexibly and quickly remove lower frequencies that we choose to remove.⁴

HP filter⁵ The reasoning behind the HP filter is as follows. Let $y_t = \log x_t$, for $t = 1, 2, \dots, T$, denote the logarithms of a time series variable x_t . The series x_t , is made up of a trend component, denoted by $\bar{y}_t = \log \bar{x}_t$, and a cyclical component, denoted by c_t , such that

$$x_t = \bar{y}_t + c_t.$$

The log in the above formulation allows us to focus attention on the normalized percentage deviations from trend, rather than less meaningful absolute deviations. It follows from the following approximation using the Taylor's theorem:

$$\begin{aligned} c_t &\equiv y_t - \bar{y}_t \equiv \log x_t - \log \bar{x}_t = \\ &= \frac{x_t - \bar{x}_t}{\bar{x}_t} + o(x_t - \bar{x}_t), \end{aligned}$$

⁴An alternative method would be to use the band-pass filter or linear detrending. The advantage of the band-pass filter is that the frequency domain can be precisely defined, and in the limit the filter exactly cuts off the frequencies we want to abstract from. However, since this is only a limiting result, the transparency of the HP filter smoothing still makes a preferred choice. For more details on band-pass filter, see Lawrence J. Christiano & Terry J. Fitzgerald, 1999. "The Band Pass Filter," NBER Working Papers 7257, National Bureau of Economic Research, Inc.

⁵You can find a free HP filter for MATLAB at: <http://dge.repec.org/codes/izvorski/hpfilter.m>

where $\bar{x}_t \equiv \exp(\bar{y}_t)$.

Given an adequately chosen positive value of λ , the HP-trend component \bar{y}_t solves

$$\min_{\{\bar{y}_t\}_t^T} \sum_t^T (y_t - \bar{y}_t)^2 + \lambda \sum_t^{T-1} [(\bar{y}_{t+1} - \bar{y}_t) - (\bar{y}_t - \bar{y}_{t-1})]^2.$$

The above objective function can be understood as follows. The first term is the sum of the squared deviations $y_t - \bar{y}_t$ which penalizes the cyclical component. So, the trend component should be as close as possible to the actual series. The second term is a multiple λ of the sum of the squares of the trend component's second differences. This term penalizes variations in the growth rate of the trend component. By putting the two terms together, the objective function trades off the smoothness of the trend (second component) with the objective of tracking closely the actual time-series. Clearly, the larger the value of λ , the more weight we put on trend being smooth. Hodrick and Prescott advise that, for quarterly data, a value of $\lambda = 1600$ is a reasonable. For annual data, a value of 6.25 is recommended⁶, although in the case of annual data researchers often use values significantly higher than that.

From now on, unless otherwise noted, all statistics we talk about refer to variables that are first logged and the HP filtered to recover the cyclical component. Occasionally, we will use linear detrending to preserve the information about persistence of the underlying time-series.

2.3 Prototype International Business Cycle Model

In this section, we set up a simple two-country endowment model (along the lines of Lucas (1982)). The model will adopt the Armington framework as the underlying trade model, and in line with the RBC literature assume that the business cycles are driven by an assumed stochastic process. Agents in the model will be aware of the

⁶Ravn, Morten O. & Uhlig, Harald, 2001. "On Adjusting the HP-Filter for the Frequency of Observations," CEPR Discussion Papers 2858, C.E.P.R. Discussion Papers.

properties of this process and how the economy works.⁷

Our simple model will allow us to focus attention on the workings of the demand-side of business cycle models (including some NOEM models), and for the time being will ignore the supply side. This will be a very useful exercise to develop intuition for later.

Model Economy

There are 2 ex-ante symmetric countries labeled domestic country, and foreign country. Households in the domestic country own an endowment tree that stochastically pays off in the domestic goods d , and households in the foreign country own a similar endowment tree that pays off in the foreign good f . Both types of goods, d and f , are used for consumption, are not perfectly substitutable, and are tradable. Trade is frictionless, but preferences are biased towards the local good.⁸

Uncertainty

In any period t , the world economy experiences one of the finitely many stochastic events $s_t \in S$. The history of such events up to and including period t is denoted by⁹ s^t , where $s^t = (s_0, s_1, \dots, s_t)$. The product probability of each history is known and denoted by $\pi(s^t)$.

For later use, note that our history dependent notation implies that it is the same to write (s^{t-1}, s_t) as s^t or (s_0, s_1, \dots, s_t) .

⁷Models departing from the assumption of rational expectations are not policy invariant and subject to Lucas critique. See Lucas (1976), “Econometric Policy Evaluation: A Critique”, Carnegie-Rochester Conference Series on Public Policy.

⁸Here we build trade friction directly into preferences as a home bias parameter ω . An almost equivalent formulation would be to assume an iceberg transportation cost instead.

⁹In particular, the realization of the exogenous endowment is a function of the history of events, i.e. $y(s^t), y^*(s^t)$.

Asset Market

In each country, there is a separate asset market in which a one-period forward state contingent bond is traded. At home, this bond is denominated in the home country numeraire, and abroad this bond is denominated in the foreign country numeraire. Bonds allow households to frictionlessly transfer wealth across all dates and states.

The asset market is assumed to be internationally integrated, in the sense that households in each country can trade bonds of both types bond. Specifically, there is a state contingent price of the foreign numeraire in terms of domestic numeraire that allows the households to trade foreign numeraire good for the domestic one. This price is denoted by x .¹⁰

In terms of notation, all variables that have a foreign country analog, are distinguished by an asterisk, and the setup is symmetric between the domestic and the foreign country. By symmetry we mean ex-ante symmetry, meaning that the same probability laws that govern the stochastic realization of endowment in each country.

Households

Households supply goods to the market, trade financial assets, and purchase final consumption goods. At each history s^t , they choose their allocation,

$$c(s^t), d(s^t), f(s^t), \{B_d(s_{t+1}, s^t), B_f(s_{t+1}, s^t)\}_{s_{t+1} \in S},$$

to maximize the expected present discounted utility

$$U = \max \sum_t \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c(s^t)) \quad (2.1)$$

¹⁰Two types of bonds are introduced to make the model symmetric. One state contingent bond would be sufficient (equivalent).

$$c(s^t) = G(d(s^t), f(s^t)), \text{ all } s^t \in S^t,$$

where $G(\cdot)$ is given by CES aggregator parameterized by elasticity σ and home-bias parameter ω :

$$G(d, f) = \begin{cases} d^\omega f^{1-\omega} & \text{if } \sigma = 1 \\ (\omega d^{\frac{\sigma-1}{\sigma}} + (1-\omega)f^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} & \text{if } \sigma \neq 1 \end{cases},$$

and the utility function is parameterized by risk aversion parameter θ :

$$u(c) = \begin{cases} \log(c) & \text{if } \theta = 1 \\ \frac{c^{1-\theta}}{1-\theta} & \text{if } \theta \neq 1 \end{cases}.$$

Household's utility maximization is subject to: (i) non-Ponzi condition $B_d(s_{t+1}, s^t) \geq -B$, $B_f(s_{t+1}, s^t) \geq -B$, where B is some arbitrarily large constant, (ii) non-negativity for all variables except for bond holdings, and (ii) the budget constraint,

$$p_d(s^t)d(s^t) + p_f(s^t)f(s^t) + \tag{2.2}$$

$$+ \sum_{s_{t+1} \in S} Q(s_{t+1}|s^t)B_d(s_{t+1}, s^t) + \tag{2.3}$$

$$+ \sum_{s_{t+1} \in S} x(s^t)Q^*(s_{t+1}|s^t)B_f(s_{t+1}, s^t) \tag{2.4}$$

$$= B_d(s^t) + x(s^t)B_f(s^t) + p_d(s^t)y(s^t), \text{ all } s^t.$$

The budget constraint reads from left to right: (i) expenditures on consumption $p_d(s^t)d(s^t) + p_f(s^t)f(s^t)$, (ii) purchases of state contingent domestic bonds $B_d(s_{t+1}, s^t)$ at price $Q(s_{t+1}|s^t)$, (ii) purchases of a set of state contingent foreign bonds $B_f(s_{t+1}, s^t)$ at price $x(s^t)Q^*(s_{t+1}|s^t)$ (Q^* is denominated in the foreign numeraire units and must be translated to domestic numeraire units through x), (iii) income from maturing domestic bonds that have been purchased at s^{t-1} for contingency s_t , (iii) income from

maturing foreign bonds purchased at state s^{t-1} for contingency s_t , and (iv) the endowment income $p_d(s^t)y(s^t)$.

The foreign households solve an analogous problem, which we state below to clarify the notation:

$$U^* = \max \sum_t^\infty \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c^*(s^t)) \quad (2.5)$$

$$c^*(s^t) = G(f^*(s^t), d^*(s^t)), \text{ all } s^t \in S^t$$

subject to

$$\begin{aligned} & p_d^*(s^t)d^*(s^t) + p_f^*(s^t)f^*(s^t) + \\ & + \sum_{s_{t+1} \in S} Q^*(s_{t+1}|s^t) B_f^*(s_{t+1}, s^t) + \\ & + \sum_{s_{t+1} \in S} \frac{Q(s_{t+1}|s^t)}{x(s^t)} B_d^*(s_{t+1}, s^t) \\ & = \frac{B_d^*(s^t)}{x(s^t)} + B_f^*(s^t) + p_f^*(s^t)y^*(s^t), \text{ all } s^t \in S^t \end{aligned} \quad (2.6)$$

The implied price index for the domestic country is given by

$$P(s^t) = \min_{G(d(s^t), f(s^t))=1} [p_d(s^t)d(s^t) + p_f(s^t)f(s^t)] \quad (2.7)$$

and for the foreign country by

$$P^*(s^t) = \min_{G(f^*(s^t), d^*(s^t))=1} [p_d^*(s^t)d^*(s^t) + p_f^*(s^t)f^*(s^t)] . \quad (2.8)$$

Later, we will find it convenient to normalize the prices by assuming that the composite consumption baskets in each country are the numeraire (composite consumption

basket will serve as numeraire in each country), i.e.

$$P(s^t) = P^*(s^t) = 1, \text{ all } s^t. \quad (2.9)$$

Under such numeraire normalization, by definition, the domestic bond pays off in the domestic composite consumption, and the foreign bond pays off in the foreign composite consumption. The relative price of the foreign numeraire in terms of the domestic numeraire $x(s^t)$ is then, again by definition, the ideal¹¹ real exchange rate. *Real exchange rate is defined as the price of the foreign consumption basket in terms of the domestic consumption basket.* In the data, the real exchange rate is measured by the ratio of foreign CPI to domestic CPI measured in common unit, $x = \frac{eCPI^*}{CPI}$, where $eCPI^*$ is foreign CPI converted to home currency units using nominal exchange rate e . Because in the model weights of the CPI price index are optimal, we refer to it as *ideal* CPI, and *ideal* real exchange rate. Sometimes this distinction of theoretical real exchange rate and a corresponding data object makes a difference (like in the model by Ghironi and Melitz (2005)). Here, the ideal CPI and fixed-weights CPI are almost the same thing, and we will not make an explicit distinction.

Feasibility and Market Clearing

Finally, the allocation must fulfil several market clearing and feasibility conditions. First, frictionless trade in the goods market requires that the law of one price holds:¹²

$$\begin{aligned} p_d(s^t) &= x(s^t)p_d^*(s^t), \\ p_f(s^t) &= x(s^t)p_f^*(s^t), \text{ all } s^t. \end{aligned} \quad (2.10)$$

¹¹Ideal (sometimes also called welfare-based) means that the basket with respect to which CPI is measured is optimal on a period by period basis.

¹²We could have made the above condition endogenous by building in a choice about location of sales into the household's utility maximization problem.

Second, supply of goods must equal demand

$$\begin{aligned} d(s^t) + d^*(s^t) &= y(s^t), \\ f(s^t) + f^*(s^t) &= y^*(s^t), \text{ all } s^t. \end{aligned} \tag{2.11}$$

Third, assets must be in zero-net supply:

$$\begin{aligned} B_d(s_{t+1}, s^t) + B_d^*(s_{t+1}, s^t) &= 0, \\ B_f(s_{t+1}, s^t) + B_f^*(s_{t+1}, s^t) &= 0, \text{ all } s^t. \end{aligned} \tag{2.12}$$

Definition of Equilibrium

Having formally laid out the model economy, we are now ready to define the equilibrium.

Definition 38 *By **competitive equilibrium** in this economy, we mean:*

- *prices $p_d(s^t), p_f(s^t), p_d^*(s^t), p_f^*(s^t), Q(s_{t+1}|s^t), Q^*(s_{t+1}|s^t), x(s^t),$*
- *and allocation $d(s^t), f(s^t), d^*(s^t), f^*(s^t), c(s^t), c^*(s^t), B_d(s_{t+1}|s^t),$*
 $B_d^*(s_{t+1}|s^t), B_f(s_{t+1}|s^t), B_f^*(s_{t+1}|s^t)$

such that

- *given prices, allocation solves household's problem given by (2.22) and (2.5),*
- *law of one price (2.10) is satisfied,*
- *and all markets clear.*

Recursive Formulation

[to be completed]

Characterization of the Equilibrium

Note that because asset markets are complete, and the households can independently transfer wealth between any states and dates, the 1st Welfare Theorem applies. Consequently, the equilibrium allocation is Pareto optimal, and thus solves a planning problem.

Given that the households are ex-ante symmetric, and so their ex-ante wealth is the same, the planning problem for this economy is to choose the allocation to maximize:

$$\max \left[\sum_t \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c(s^t)) + \sum_t \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c^*(s^t)) \right]$$

subject to aggregation constraints

$$\begin{aligned} c(s^t) &= G(d(s^t), f(s^t)), \\ c^*(s^t) &= f^*(s^t)^\omega d^*(s^t)^{1-\omega}, \text{ all } s^t \in S^t \end{aligned}$$

and and feasibility constraints

$$\begin{aligned} d(s^t) + d^*(s^t) &= y(s^t), \\ f(s^t) + f^*(s^t) &= y^*(s^t), \text{ all } s^t, \end{aligned}$$

Since in the above problem, periods are not physically connected through the objective function of the constraint set (no state variables, objective function is time-separable), wlog, we can recast the above dynamic problem as a sequence of static

planning problems given by:

$$\max_{c, c^*, d, f} [u(c(s^t)) + u(c^*(s^t))]$$

subject to

$$c(s^t) = G(d(s^t), f(s^t)),$$

$$c^*(s^t) = G(f^*(s^t), d^*(s^t)),$$

and

$$d(s^t) + d^*(s^t) = y(s^t),$$

$$f(s^t) + f^*(s^t) = y^*(s^t), \text{ all } s^t.$$

Since the above planning problem has a unique solution, the 2nd Welfare Theorem implies that the CE allocation not only exists, but it is unique.

Exercise 39 *Prove the equivalence between dynamic problem and a sequence of static problems formally. HINT: You are asked to show that if an allocation solves one problem, it solves the other.*

Exercise 40 *Assume $\theta = 1$ and $\sigma = 1$. Show that the solution to the above planning*

problem is of the form:

$$\begin{aligned}
 c(s^t) &= \omega^\omega (1 - \omega)^{1-\omega} y(s^t)^\omega y^*(s^t)^{1-\omega}, \\
 c^*(s^t) &= \omega^\omega (1 - \omega)^{1-\omega} y^*(s^t)^\omega y(s^t)^{1-\omega} \\
 d(s^t) &= \omega y(s^t), \\
 d^*(s^t) &= (1 - \omega) y(s^t), \\
 f(s^t) &= (1 - \omega) y^*(s^t), \\
 f^*(s^t) &= \omega y^*(s^t), \text{ all } s^t.
 \end{aligned}$$

Exercise 41 Set $\theta = 1$ and $\sigma = 1$, and consider financial autarky (no borrowing and lending possible), i.e.

$$B_d(s_{t+1}|s^t) = B_d^*(s_{t+1}|s^t) = B_f^*(s_{t+1}|s^t) = B_f(s_{t+1}|s^t) = 0, \text{ all } s^t, s_{t+1}.$$

Show that the allocation under financial autarky is exactly the same as in the complete asset market economy. Interpret your findings. In particular, answer what it implies about the dynamics of the trade balance in this economy.

We next proceed to find the supporting prices and complete the characterization of the equilibrium.

Prices

Let $\lambda(s^t)$ be the multiplier on budget constraint, and $\mu(s^t)$ be the multiplier on the aggregation constraint.

The first order conditions of the domestic household are given by

$$\begin{aligned}
c(s^t) &: \beta\pi(s^t)u'(c(s^t)) - \mu(s^t) = 0, \\
B(s_{t+1}|s^t) &: -\lambda(s^t)Q(s_{t+1}|s^t) + \lambda(s^{t+1}) = 0, \\
B^*(s_{t+1}|s^t) &: -\lambda(s^t)x(s^t)Q^*(s_{t+1}|s^t) + \lambda(s^{t+1})x(s^{t+1}) = 0, \\
d(s^t) &: -\mu(s^t)G_d(d(s^t), f(s^t)) - \lambda(s^t)p_d(s^t) = 0, \\
f(s^t) &: -\mu(s^t)G_f(d(s^t), f(s^t)) - \lambda(s^t)p_f(s^t) = 0.
\end{aligned}$$

and the first order conditions of the foreign household are

$$\begin{aligned}
c(s^t) &: \beta\pi(s^t)u'(c^*(s^t)) - \mu^*(s^t) = 0, \\
B_f^*(s_{t+1}|s^t) &: -\lambda^*(s^t)Q^*(s_{t+1}|s^t) + \lambda^*(s^{t+1}) = 0, \\
B_d^*(s_{t+1}|s^t) &: -\lambda^*(s^t)\frac{Q(s_{t+1}|s^t)}{x(s^t)} + \frac{\lambda^*(s^{t+1})}{x(s^{t+1})} = 0, \\
d(s^t) &: -\mu^*(s^t)G_d(f(s^t), d(s^t)) - \lambda^*(s^t)p_d^*(s^t) = 0, \\
f(s^t) &: -\mu^*(s^t)G_f(f(s^t), d(s^t)) - \lambda^*(s^t)p_f^*(s^t) = 0.
\end{aligned}$$

From the first set of equations, we derive

$$\begin{aligned}
(i) &: Q(s_{t+1}|s^t) = \beta\pi(s_{t+1}|s^t)\frac{u'(c(s^{t+1}))}{u'(c(s^t))}, \\
(ii) &: \frac{x(s^t)}{x(s^{t+1})}Q^*(s_{t+1}|s^t) = \beta\pi(s_{t+1}|s^t)\frac{u'(c(s^{t+1}))}{u'(c(s^t))}, \\
(iii) &: p_d(s^t) = G_d(d(s^t), f(s^t)), \\
(iv) &: p_f(s^t) = G_f(d(s^t), f(s^t)).
\end{aligned} \tag{2.13}$$

where $\pi(s_{t+1}|s^t) \equiv \frac{\pi(s^{t+1})}{\pi(s^t)}$ is the conditional probability of state s_{t+1} conditional on

s^t , and from the second set, we have:

$$\begin{aligned}
 (v) \quad & Q^*(s_{t+1}|s^t) = \beta \pi(s_{t+1}|s^t) \frac{u'(c^*(s^{t+1}))}{u'(c^*(s^t))}, \\
 (vi) \quad & \frac{x(s^{t+1})}{x(s^t)} Q(s_{t+1}|s^t) = \beta \pi(s_{t+1}|s^t) \frac{u'(c^*(s^{t+1}))}{u'(c^*(s^t))}, \\
 (vii) \quad & p_d^*(s^t) = G_d(f^*(s^t), d^*(s^t)), \\
 (viii) \quad & p_f(s^t) = G_f(f^*(s^t), d^*(s^t)).
 \end{aligned} \tag{2.14}$$

Note that if prices are normalized so that $P \equiv P^* \equiv 1$, we must have

$$\lambda(s^t) = \mu(s^t).$$

Exercise 42 Show that $\lambda(s^t) = \mu(s^t)$. *HINT: Use the fact that at the optimal solution, by definition of the price index, we have:*

$$p_d(s^t)d(s^t) + p_f(s^t)f(s^t) = P(s^t)c(s^t) = P(s^t)d(s^t)^\omega f(s^t)^{1-\omega}.$$

Exploit the equality to show $\mu = \lambda$.

Uncovered Interest Rate Parity In the first step, we combine equations (i) and (ii) to obtain a state-by-state non-arbitrage condition on bond prices:

$$\frac{x(s^{t+1})}{x(s^t)} = \frac{Q^*(s_{t+1}|s^t)}{Q(s_{t+1}|s^t)}. \tag{2.15}$$

Intuitively, this condition says that there are no pure profits taking a short position on one bond in some states and an offsetting long position on the other bond.

The above condition is closely related to the so called *uncovered interest rate parity*. The *uncovered interest rate parity* states that the domestic risk-free interest rate must be equal to the foreign risk free interest rate augmented by the expected

change of the exchange rate between the two periods. In real terms, if we let r denote an effective¹³ risk free domestic interest rate on domestic bonds and r^* denote the effective risk-free interest rate on foreign bonds, the uncovered interest rate parity says:

$$\exp(r(s^t)) = E[\exp(r^*(s^t)) \frac{x(s^{t+1})}{x(s^t)}],$$

which we can rewrite in logs as

$$r(s^t) = r^*(s^t) + \log E(\frac{x(s^{t+1})}{x(s^t)} | s^t).$$

Since UIP equalizes the expected payoff from the two alternative investment strategies that shift wealth from today to tomorrow, when agents are risk neutral and rational, it should be clear why such condition must hold. In particular, in such case, it follows directly from condition (2.15) listed above—but not all the way around.

However, when the agents are risk averse, the state-by-state non-arbitrage condition turns out to be not equivalent to the UIP condition. The reason is that investment in the home risk-free bond is risk free for the home households, but investment in the foreign bond is not due to stochastic movements of the real exchange rate that may be potentially correlated with consumption. Thus, in general, the model may not or may predict that UIP should hold. In what follows, we will investigate to what extent it does.

Before we proceed, it is convenient to adopt the standard language of finance and define the conditional pricing kernels¹⁴ (called also stochastic discount factor SDF)

¹³The definition of an effective return of a bond is $r = -\log P$, where P is the per-unit price of the bond. The formula comes from the idea that compounding is continuous, and so the future value of the price of the bond under continuous compounding must be equal to the promised payoff of the bond ($=\$1$): $1\$ = \lim_{n \rightarrow \infty} P(1 + \frac{r}{n})^n = P \exp(r)$, and so $r = -\log P$.

¹⁴Pricing kernel is a price of one unit of payoff in state s_{t+1} (following history s^t) under an abstract assumption that state s_{t+1} occurs with probability 1.

that isolate the probabilities from the state contingent prices Q and Q^* . In our model, they are given by

$$\begin{aligned} M(s_{t+1}|s^t) &= \frac{Q(s_{t+1}|s^t)}{\pi(s_{t+1}|s^t)}, \\ M^*(s_{t+1}|s^t) &= \frac{Q^*(s_{t+1}|s^t)}{\pi(s_{t+1}|s^t)}. \end{aligned}$$

The reason why it is more convenient is because using the conditional pricing kernels we can simply price assets using an expectation operator, and that simplifies notation.¹⁵ For example, in the case of one-period forward assets, we can price it as follows:

$$(\text{Price of asset}) = E[M(s_{t+1}|s^t) \times (\text{payoff of asset at } s^{t+1})|s^t],$$

which can be written as

$$(\text{Price of asset}) = E_t[M \times (\text{payoff of asset at } t + 1)].$$

Moreover, dividing both sides of the above equation by the price of the asset, we can write

$$1 = E_t[M \times (\text{implied return on asset at } t + 1)].$$

In equilibrium, such condition must hold for all assets that are traded (with no frictions). Otherwise, the risk averse agent, with kernel M , could profit from arbitrage.

Going back to our model, we let r be the effective return on a one period forward risk-free domestic bond denominated in domestic composite consumption, and we let r^* to be the effective return on a one period forward risk free foreign bond (denom-

¹⁵It also makes the applications of probabilistic calculus more straightforward (without going through the integrals, we can use the formulas readily available).

inated in foreign consumption). Given pricing kernels defined above, in equilibrium, the domestic household must be indifferent whether to invest in any of the two assets, and so:

$$\begin{aligned} 1 &= E_t[M \times \frac{x_{t+1} \exp(r_t^*)}{x_t}], \\ 1 &= E_t[M \times \exp(r_t)]. \end{aligned}$$

(The above conditions would be endogenously implied by FOC if the budget constraint additionally included risk-free assets in the household problem.)

Combining the above asset pricing equations, we obtain

$$E_t[M \times \exp(r_t)] = E_t[M \times \frac{x_{t+1} \exp(r_t^*)}{x_t}],$$

which in log terms implies

$$r_t - r_t^* = \log E_t[M \times \frac{x_{t+1}}{x_t}] - \log E_t[M].$$

Using (2.15), we next note that

$$\log E_t[M \times \frac{x_{t+1}}{x_t}] = \log E_t[M^*].$$

Substituting into the previous expression, we derive

$$r_t - r_t^* = \log E_t[M^*] - \log E_t[M], \quad (2.16)$$

which in combination with the expression derived from taking expectation of the log of equation (2.15),

$$E_t \log \frac{x(s^{t+1})}{x(s^t)} = E_t[\log M^*] - E_t[\log M], \quad (2.17)$$

gives

$$\begin{aligned}
 r_t - r_t^* &= \log E_t[M^*] - \log E_t[M] = \\
 &= \log E_t[M^*] - \log E_t[M] + \\
 &\quad + \left[E_t \left[\log \frac{x(s^{t+1})}{x(s^t)} \right] - E_t[\log M^*] + E_t[\log M] \right].
 \end{aligned}$$

Defining the residual as *risk premium* \mathcal{P}_t (risk premium essentially means residual)

$$\begin{aligned}
 \mathcal{P}_t &\equiv [\log E_t[M^*] - E_t[\log M^*]] - [\log E_t[M] - E_t[\log M]] + \\
 &\quad + E_t \log \frac{x(s^{t+1})}{x(s^t)} - \log E_t \frac{x(s^{t+1})}{x(s^t)}
 \end{aligned}$$

we can write the UIP condition implied by the the model as

$$r_t = r_t^* + \log E_t \frac{x(s^{t+1})}{x(s^t)} + \mathcal{P}_t.$$

To understand the intuition behind the above equation, it is instructive to consider a special case of pricing kernels and real exchange rate growth rate that are log-normally distributed. The trick here is that in the case of a lognormally distributed random variable, we can easily evaluate the expectation of this variable by exploiting the following fact:

$$X \sim \log \text{ normal} \tag{2.18}$$

$$E(X) = e^{\mu + \sigma^2/2},$$

where μ is the mean of $\log X$ (which is normally distributed), and σ^2 is the variance of $\log X$.

Exercise 43 *Prove the above property by integrating over the normal distribution.*

So, let's assume that $\log M$, $\log M^*$ and $\log \frac{x_{t+1}}{x_t}$ are both normally distributed. In such case, from (2.18), we have

$$\begin{aligned}\log E_t[M] &= \log e^{E_t[\log M] + \text{Var}_t[\log M]/2}, \\ \log E_t[M^*] &= \log e^{E_t[\log M^*] + \text{Var}_t[\log M^*]/2}, \\ \log E_t\left[\frac{x_{t+1}}{x_t}\right] &= \log e^{E_t\left[\log \frac{x_{t+1}}{x_t}\right] + \text{Var}_t\left[\log \frac{x_{t+1}}{x_t}\right]/2}\end{aligned}$$

thus

$$\mathcal{P}_t = \frac{\text{Var}_t[\log M^*]}{2} - \frac{\text{Var}_t[\log M]}{2} - \frac{\text{Var}_t\left[\log \frac{x_{t+1}}{x_t}\right]}{2},$$

and

$$r_t = r_t^* + \log E_t\left[\frac{x(s^{t+1})}{x(s^t)}\right] + \mathcal{P}_t, \quad (2.19)$$

where E_t and Var_t denote conditional expectation based on the information available at t and conditional variance based on the information available at t , respectively.

The above simplified equation is much easier to interpret and understand. It simply says that the return on the domestic risk free bond adds a premium (potentially time-varying) whenever the difference between the conditional volatility of the foreign pricing kernel and domestic pricing kernel differs, or alternatively conditional variance of the real exchange rate growth differs.

This above result can be understood as follows. Since the depreciation of the real exchange rate is correlated with the domestic pricing kernel, we should expect that either long- or short position in the foreign bond market is a good hedge for domestic households against their consumption risk. For the sake of argument, say that a long position on the foreign bond hedges home households.¹⁶ Consequently, in order to hedge consumption risk, the domestic country household could take a short position on the domestic country bond and a long position on the foreign country bond.

¹⁶What this means is that the pricing kernel M positively covaries with the payoff of the bond $\frac{x_{t+1}}{x_t} r^*$, and so the bond pays off exactly when the household needs more consumption (is hungry).

Similarly, the foreign country household could take a long position on the domestic country bond and a short position on the foreign country bond. Now, because these position would be exactly offsetting, as long as there is symmetry in consumption risk across countries, the market for bonds can clear, and neither bond is traded at a premium. However, there is a flip side to this argument. Whenever symmetry is distorted in some way, so that one of the countries faces higher consumption risk as measured by $var_t(M)$, the demand and supply for the bonds will no longer be balanced. As a result, one of the bonds will have to be traded at a premium for the market to clear. This is exactly what the risk premium term \mathcal{P}_t captures.

(The second ‘real exchange rate’ term is just a mechanical implication of Jensen inequality and volatility of real exchange rate, and doesn’t play a major role in deviations from UIP in the data.)

Exercise 44 *Assume, you know the state contingent effective rate of inflation both at home and abroad. Let the notation for inflation be $\Pi(s^t)$, and $\Pi^*(s^t)$, to distinguish inflation from the state-contingent probabilities. Assuming log-normality whenever needed, derive the deviations from the nominal version of the UIP in our model. HINT: Real exchange rate is linked to nominal exchange rate e by the following relation $x = \frac{eP^*}{P}$, and so $\frac{x_{t+1}}{x_t} = \frac{e_{t+1}}{e_t} \frac{\exp(\Pi^*(s^t))}{\exp(\Pi(s^t))}$.*

Forward Premium Puzzle Despite its theoretical appeal and simplicity, there is little evidence supporting the uncovered interest rate parity in the data. The UIP relation has been tested widely by running a regression of interest rate premia $r - r^*$ on the real exchange rate changes (called Fama regression). According to the UIP hypothesis, the implied regression coefficient should be positive, and close to 1. The puzzling finding is that it is significantly negative and its value is around¹⁷ -3, which is referred to as the *forward premium anomaly or the UIP puzzle*.

¹⁷See the survey by Engel (1996).

Clearly, from the perspective of a risk neutral agent, the failure of the UIP relation implies that there are excess returns from taking a short position on the low interest rate currencies (like Japan), and by taking an offsetting long position on the high interest rate currencies (like Poland). This strategy is called ‘carry trade’, and it is actually exploited by investors (and they do make money!). Of course, this does not necessarily imply that the ‘carry trade’ strategy is a good deal for a risk averse agent, as the historically observed premium may be simply a compensation for risk. The literature has tried to identify the risk factor associated with the forward premium, and has so far failed to identify one.

The question is whether our model, which does imply some deviations from UIP, can account for the negative regression coefficient. Unfortunately, the answer is negative. In our model, the term P_t is not going to move much, and UIP will approximately hold.¹⁸ Our next task is to demonstrate this property of the model.

To this end, let’s take a look at the analytical case with log utility and Cobb-Douglas utility function. In this case, we know that

$$\begin{aligned} Var_t [\log M] &= Var_t \left[\log \beta \frac{c_{t+1}}{c_t} \right] = \\ &= Var_t \left[\log \beta \omega^\omega (1 - \omega)^{1-\omega} \frac{y_{t+1}^\omega}{y_t} \frac{y_{t+1}^{*1-\omega}}{y_t^*} \right] = \\ &= Var_t \left[\omega \log \frac{y_{t+1}}{y_t} + (1 - \omega) \log \frac{y_{t+1}^*}{y_t^*} \right]. \end{aligned}$$

Assuming the standard AR(1) stochastic process for income,

$$\log y_{t+1} = \rho \log y_t + \varepsilon_t,$$

¹⁸It is called forward premium puzzle because the interest rate differential can be expressed as a difference between the forward exchange rate and spot exchange rate, $f_t - s_t = r - r^*$. This is a completely risk free arbitrage condition, as here the investor needs to buy a foreign bond and at the same time a future contract for the exchange rate. This relation, called covered interest rate parity, holds in the data. The term $f_t - s_t$ is called forward premium.

we calculate (note: variance is conditional on period t and so $E_t \log y_t$ is just a known constant here)

$$\begin{aligned}
 Var_t [\log M] &= Var_t \left[\omega \log \frac{y_{t+1}}{y_t} + (1 - \omega) \log \frac{y_{t+1}^*}{y_t^*} \right] = \\
 &= \omega^2 Var_t [\log y_{t+1}] + (1 - \omega)^2 Var_t [\log y_{t+1}^*] + \\
 &\quad 2\omega(1 - \omega) Cov_t [\log y_{t+1}^*, \log y_{t+1}] = \\
 &= \omega^2 Var_t [\varepsilon_{t+1}] + (1 - \omega)^2 Var_t [\varepsilon_{t+1}^*] + \\
 &\quad + 2\omega(1 - \omega) Cov_t [\varepsilon_{t+1}^*, \varepsilon_{t+1}]
 \end{aligned}$$

and thus under symmetry (i.e. $Var_t [\varepsilon_{t+1}] = Var_t [\varepsilon_{t+1}^*]$) :

$$\frac{Var_t [\log M^*]}{2} - \frac{Var_t [\log M]}{2} = 0.$$

The above finding is not just a property of the particular log/Cob-Douglas case. A similar property holds more generally. In fact, it is not an easy task to generate sensibly looking fluctuations of the risk premium in this class of models. The problem seems to be the conditioning on the information from period t that shows up in the formula for risk premium \mathcal{P}_t . To have any time-varying fluctuations, the model has to generate heteroscedasticity in either the uncertainty structure or the sensitivity of the pricing kernels to uncertainty. One of the features that can give rise to the latter are market segmentation and habit formation. This has been documented in the following papers: Atkeson, Kehoe and Alvarez (2008): “Time-Varying Risk, Interest Rates, and Exchange Rates in General Equilibrium”, Minneapolis FED Staff Report 371, September, and Verdelhan (2008): “A Habit-Based Explanation of the Exchange Rate Risk Premium”, Journal of Finance, forthcoming.¹⁹ Some authors also try to

¹⁹In the first paper mentioned above, the pricing kernel M and M^* is the pricing kernel of only active traders in a given moment of time, and exhibits time-varying sensitivity to shocks. In the second paper, the conditional variance of the kernel is time-varying due to time-varying risk aversion in the habit model. An alternative way to go, more difficult to discipline, is to play with

account for this fact by models exhibiting ambiguity aversion (robust control).

Real Exchange Rate and Risk Sharing In this environment, real exchange rates are tightly linked to relative consumption. Combining equations (i) and (vi) to substitute out for $Q(s_{t+1}, s^t)$, a recursive equation for the real exchange rate can be obtained:

$$x(s^{t+1}) = x(s^t) \frac{u'(c(s^t))}{u'(c(s^{t+1}))} \frac{u'(c^*(s^{t+1}))}{u'(c^*(s^t))}.$$

Using the fact that in state s^0 is deterministic, under the assumption of ex-ante symmetry, one can collapse the above recursive law by solving it backwards, and write instead

$$x(s^t) = \frac{u'(c^*(s^t))}{u'(c(s^t))}. \quad (2.20)$$

The above equation is probably the most famous equation in international economics. It says that households trade assets to equalize the MRS of consumption abroad to consumption at home so that it aligns with the relative price of consumption abroad to consumption at home. As a result, to the first approximation, a household consumes more in a given state and date if and only if its consumption is cheaper in this state and date.

The above condition would not be that surprising, if the MRS pertained to the same household. It would then say that the household optimally trades off consumption of c^* and c — a standard optimality condition to maximize total utility. However, the MRS here pertains to two marginal utility of two separate households in two different countries. Nevertheless, the model predicts that these households will trade assets, and effectively act as if they were a family maximizing the joint utility. In this sense, we can say that this equation represents *perfect risk sharing*, and refer to this condition as *perfect risk sharing condition*.

the uncertainty structure of the model and introduce time-varying exposure to shocks. Some work along these lines you can find in the most recent paper by Lustig, Roussanov, Verdelhan (2008).

Implications of Perfect Risk Sharing for Real Exchange Rates It is not difficult to see that our model may have a hard time to match the real exchange rate and consumption data simultaneously. The real exchange rates in the data are volatile, persistent (almost indistinguishable from random walks in short time-series), and not very correlated with anything else.²⁰ Consumption is persistent, but the least volatile among aggregate variables due to consumption smoothing.

For example, a simple model with log utility and Cob-Douglas aggregator would fall short at least by a factor of 4 in terms of volatility, presuming it accounts for the consumption data. To see this, evaluate the standard deviation of both sides of (2.20)

$$var(\log x) = var(\log(c) - \log(c^*)).$$

Using variance decomposition (denote $var(\log x) \equiv var_x$),

$$var_x = (var_c + var_{c^*} - 2corr_{c,c^*}std_c^*std_c),$$

under symmetry ($std_c = std_{c^*}$), we have:

$$\frac{std_x}{std_y} = \frac{std_c}{std_y} \sqrt{2(1 - corr_{c,c^*})}.$$

Now, for the US data (versus rest of the world)²¹, we roughly have: $\frac{std_x}{std_y} = 3$, $\frac{std_c}{std_y} = \frac{3}{4}$, and $corr_{c,c^*} = \frac{1}{4}$. Plugging in these moments to the above equation, we obtain

$$\frac{std_x}{std_y} = \frac{3}{4} \sqrt{2(1 - \frac{1}{4})} = \frac{3}{4} \sqrt{\frac{6}{4}} = \frac{3\sqrt{6}}{8} = 0.92. \quad (2.21)$$

This is about 3 times less than the value in the data.

At this point, you might be tempted to think that a mere departure from log utility to CRRA utility $u(c) = \frac{c^{1-\theta}}{1-\theta}$ should fix it. For CRRA, the analog of the above

²⁰Real exchange rates in the data closely track nominal exchange rates.

²¹See Drozd and Nosal (2008), Table 7.

condition would say:

$$var(\log x) = \sigma^2 var(\log(c) - \log(c^*)),$$

and then

$$\frac{std_x}{std_y} = \sigma \frac{std_c}{std_y} \sqrt{2(1 - corr_{c,c^*})}.$$

So, superficially, it seems that matching the data in terms of volatility, should be just a matter of picking the right value of σ . For example, for the statistics we listed above, we would need to pick

$$\sigma \simeq 3.$$

Unfortunately, the problem is much deeper than that. The problem is that models will also typically underpredict the volatility of $\frac{std_c}{std_y}$ and overstate the correlation $corr_{c,c^*}$. As a result, the same calculation but using the values implied by the models will look much worse. Moreover, in the models with explicitly set up supply-side (physical capital accumulation; labor/leisure choice), a higher value of σ will create an additional incentive to smooth consumption using the supply-side channels, and the smoother consumption will additionally offset the effect of an increased value of σ through these statistics. Consequently, the resulting models with high σ might neither match consumption data nor the real exchange rate data. The literature refers to this problem as the *real exchange rate volatility puzzle*.²²

To some extent, our simple model also suffers from the problem described above. Using the solution from exercise (40), we can plug in to the perfect risk sharing equation to derive:

$$x = \left(\frac{y}{y^*}\right)^{(2\omega-1)}.$$

Since our domestic country is US, to match the share of imports in US GDP, ω would

²²Obstfeld and Rogoff (2000) give a nice overview of the 6 main puzzles of international economics.

have to be something around .85 (this is the share of expenditures on foreign goods in total expenditures), and we would have

$$\log x = \frac{1}{2}(\log(y) - \log(y^*)).$$

Given that output is highly correlated across countries (about .4), under symmetry, we thus must have

$$\frac{std_x}{std_y} = 1 - corr_{y,y^*} = 0.6.$$

Comparing the above result to (2.21), we see that the model performs much worse than it potentially could according to (2.21). The problem is that consumption is counterfactually more correlated than output (in the data the opposite is true). Consequently, for our model to match the data, we need to make the real exchange rate at least 5 times more volatile, which is way more than required according to (2.21).

In the endowment economy, it is still possible to match real exchange rate volatility by increasing σ to 6 or 7. Nevertheless, also in this problem the relationship between σ and real exchange volatility will be far from 1 to 1. The problem is that a high value of σ creates an additional incentive to trade more intensively to further improve on consumption smoothing. (We would have to simultaneously set γ low and σ high—this can help mechanically.)

In terms of the correlation of the real exchange rate with the consumption ratio $\frac{c}{c^*}$, the model fares as bad as in terms of volatility. Namely, according to equation (2.20) implies that the correlation of the real exchange rate with consumption ratio should be one. In the data, this correlation is negative (around -0.2). The problem with correlation is known in the literature as *Backus-Smith puzzle*.²³

²³The original reference is Backus and Smith (1993). See also Corsetti, Dedola and Leduc (2007).

Terms of Trade and Export-Import Prices The model has predictions on 3 other basic international prices: (i) the real export price $x(s^t)p_d^*(s^t) = p_d(s^t)$, (ii) the real import price $p_f(s^t)$, and (iii) the terms of trade $\frac{p_f(s^t)}{x(s^t)p_d^*(s^t)}$ (price of imports in terms of exports).

In the data, we can construct these prices by looking at the aggregator deflator price of imports or exports, and divide it by the CPI to make it real.²⁴ The deflator price is defined as a ratio of the nominal value of a variable divided by the constant price value of the variable. Terms of trade is defined as the ratio of the deflator price of imports relative to the price of exports.

According to the model, all these three prices should be closely related to the terms of trade, which we will illustrate using our simple log/Cobb-Douglas case. These properties, however, hold more generally.

In the Cobb-Douglas case, the ideal CPI can be expressed as a weighted average of the prices of good d and f , i.e.²⁵

$$P = p_d^\omega p_f^{1-\omega}.$$

and abroad

$$P^* = p_f^{*\omega} p_d^{*1-\omega}.$$

Given that the CPI is a numeraire here ($P \equiv 1$), we can derive the real export price from the following evaluation

$$p_x \equiv \frac{x p_d^*}{P} = \frac{p_d}{p_d^\omega p_f^{1-\omega}} = \left(\frac{p_f}{p_d}\right)^{\omega-1},$$

²⁴In the data, weights in the CPI are kept constant for a couple of years and do not represent optimal weights at all times. Fixing weights to measure CPI in the model has almost no effect on the time-series of the CPI.

²⁵Precisely, $P = \omega^{-\omega} (1-\omega)^{-(1-\omega)} p_d^\omega p_f^{1-\omega}$

and the real import price from

$$p_m \equiv \frac{p_f}{P} = \frac{p_f}{p_d^\omega p_f^{1-\omega}} = \left(\frac{p_f}{p_d}\right)^\omega.$$

As we can see, both prices are tightly linked to the terms of trade, which is defined as:

$$p \equiv \frac{p_f}{xp_d^*} = \frac{p_f}{p_d}.$$

Furthermore, the real exchange in this environment is also intimately linked to the terms of trade, and thus to the other prices. By definition, the real exchange rate is given by the ratio of the foreign CPI to the domestic CPI measured in a common unit. By the law of one price and assumed by us numeraire normalization ($P^* \equiv P \equiv 1$), we obtain the real exchange rate from the following evaluation:

$$x \equiv \frac{xP^*}{P} = \frac{x(p_f^{*\omega} p_d^{*1-\omega})}{p_d^\omega p_f^{1-\omega}} = \frac{p_f^\omega p_d^{1-\omega}}{p_d^\omega p_f^{1-\omega}} = \left(\frac{p_f}{p_d}\right)^{2\omega-1}.$$

On the basis of the above formulas, we thus conclude that under the conditions of home-bias (i.e. $\frac{1}{2} < \omega < \frac{1}{2}$), the model has very sharp predictions how these 3 aggregate price should move. First, the correlation between the real export price and the real import price should be -1, second, the correlation of real export price with the real exchange rate should be -1, and third, the terms of trade should be *more* volatile than the real exchange rate.

Export-Import Price Correlation Puzzle As we can see from Tables 2.1-2.2 and Figure 2.2), none of these predictions listed above are consistent with the data for any of the 12 countries in our sample. In fact, the opposite is true in the data. The real export and the real import prices are highly positively correlated, and the

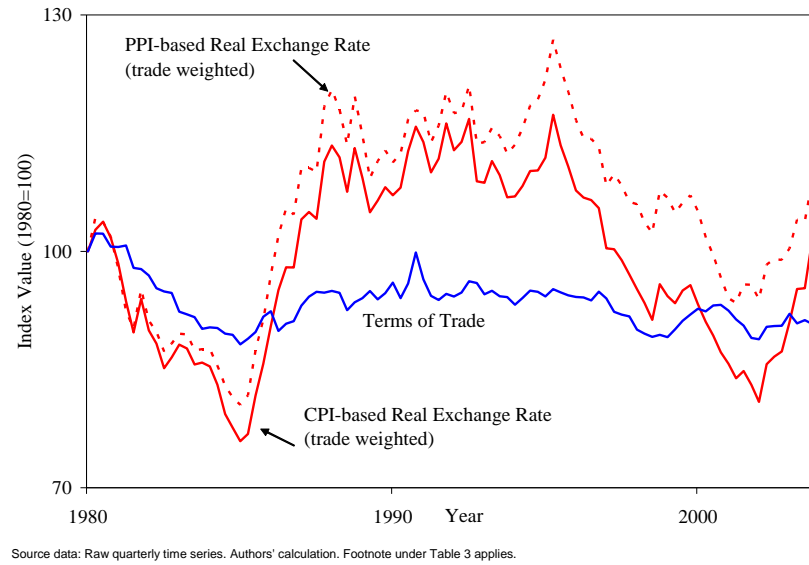


Figure 2.2: Comparison of real exchange rate with terms of trade (linearly detrended data).

terms of trade is much *less* volatile than the real exchange rate.²⁶ All prices, pretty much move with the real exchange rate. Following Drozd and Nosal (2008), we refer to these problems as: (i) *export-import price correlation puzzle* and (ii) *terms of trade relative volatility puzzle*.

Summary We summarize all our findings for prices in Table 2.3. (We should stress that these are all predictions that the corresponding closed economy version of the model would not have.)

²⁶There are reasons to actually argue that the terms of trade in the model severely understates the volatility of the terms of trade in the data. In the data, crude oil enters asymmetrically into import price, and by being very volatile price, increases volatility of the terms of trade. Since there is no crude oil in the model, we should remove it from the data and then compare the volatilities. In such case, the volatility of the terms of trade relative to the real exchange rate falls further by about 50% wrt to Table X.

Table 2.1: Real export and real import prices.s^a

| Country | Correlation | | | | Relative volatility ^b to x (%) | | |
|-------------|-------------|----------|----------|--------|---|-------|------|
| | p_x, p_m | p_x, x | p_m, x | p, x | p_x | p_m | p |
| Belgium | 0.93 | 0.71 | 0.73 | 0.43 | 105.3 | 131.7 | 50.5 |
| Canada | 0.77 | 0.55 | 0.90 | 0.56 | 75.8 | 79.4 | 52.8 |
| Switzerland | 0.62 | 0.50 | 0.85 | 0.77 | 38.2 | 83.0 | 66.5 |
| France | 0.90 | 0.63 | 0.66 | 0.59 | 74.5 | 149.9 | 89.2 |
| Germany | 0.58 | 0.44 | 0.85 | 0.83 | 33.1 | 101.4 | 86.6 |
| Italy | 0.87 | 0.65 | 0.69 | 0.60 | 55.9 | 116.5 | 73.3 |
| Japan | 0.88 | 0.92 | 0.88 | 0.72 | 38.8 | 86.1 | 55.3 |
| Netherlands | 0.94 | 0.77 | 0.81 | 0.15 | 130.9 | 134.1 | 47.0 |
| US | 0.74 | 0.39 | 0.71 | 0.67 | 37.0 | 58.6 | 40.0 |
| Australia | 0.45 | 0.32 | 0.94 | 0.83 | 42.1 | 70.4 | 63.8 |
| Sweden | 0.88 | 0.58 | 0.71 | 0.55 | 58.7 | 76.1 | 37.1 |
| UK | 0.90 | 0.59 | 0.78 | 0.64 | 58.4 | 69.1 | 30.6 |
| MEDIAN | 0.87 | 0.58 | 0.80 | 0.62 | 57.1 | 84.6 | 54.1 |

^a p_x, p_m denote real export and import prices, $p \equiv \frac{p_m}{p_x}$ denotes terms of trade, x trade weighted real exchange rate. Statistics based on logged & HP filtered quarterly series for the period 1980:1-2000:1. Data sources listed in the Appendix.

^bRelative volatility is the standard deviation relatively to the standard deviation of the country's real exchange rate

2.4 Basic Supply-Side Extensions

This section extends the basic setup we have discussed above to incorporate production and labor-leisure choice along the lines of Backus, Kehoe and Kydland (1995) model (BKK hereafter).

At this point, it is important to stress that capital accumulation does not invalidate any of the findings from the previous section. Capital accumulation and production both pertain to the upstream structure of the model, and since we have not relied upon the particular properties of the endowment process, our results still stand. (Labor-leisure choice may invalidate some of our finding, but it does not. We will demonstrate it later quantitatively.)

Table 2.2: Relative volatility of the terms of trade.

| Country | Volatility of p relative to x (in %) | | |
|-------------|--|------------|----------------|
| | Price index used to construct ^a x | | |
| | CPI all-items | WPI or PPI | None (nominal) |
| Australia | 0.51 | 0.54 | 0.60 |
| Belgium | 0.57 | 0.70 | 0.47 |
| Canada | 0.56 | 0.76 | 0.61 |
| France | 0.80 | 0.74 | 0.73 |
| Germany | 0.83 | 0.81 | 0.80 |
| Italy | 0.75 | 0.79 | 0.77 |
| Japan | 0.52 | 0.54 | 0.55 |
| Netherlands | 0.52 | 0.49 | 0.44 |
| Sweden | 0.21 | 0.21 | 0.37 |
| Switzerland | 0.71 | 0.68 | 0.67 |
| UK | 0.30 | 0.32 | 0.37 |
| US | 0.31 | 0.33 | 0.28 |
| MEDIAN | 0.54 | 0.61 | 0.57 |

Notes: We have constructed trade-weighted exchange rates using weights and bilateral exchange rates for the set of 11 fixed trading partners for each country. The trading partners included in the sample are the countries listed in this table. Statistics are computed from logged and H-P-filtered quarterly time-series for the time period 1980:1-2000.01 ($\lambda = 1600$). Data sources are listed at the end of the paper.

^aRER constructed these indices instead of the CPI.

Physical Capital Accumulation

To incorporate investment, capital and production to our economy, we must modify the household's problem to include capital accumulation decision and split up income into various factor payments. The modified problem of the household is given by:

$$U = \max \sum_t \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c(s^t)) \quad (2.22)$$

subject to

$$\begin{aligned} c(s^t) + i(s^t) &= G(d(s^t), f(s^t)), \\ gk(s^{t+1}) &= (1 - \delta)k(s^t) + i(s^t), \end{aligned}$$

Table 2.3: Summary of the puzzles on prices.

| Statistic | Data | Model |
|--------------------------|-----------------|-------------|
| UIP coefficient | around -3 | around 1 |
| $std(x)/std(y)$ | between 3 and 6 | less than 1 |
| $corr(x, \frac{c}{c^*})$ | negative | +1 |
| $corr(p_x, p_m)$ | highly positive | -1 |
| $corr(p_x, x)$ | highly positive | -1 |
| $std(p)/std(x)$ | less than 1 | more than 1 |

and

$$\begin{aligned}
& p_d(s^t)d(s^t) + p_f(s^t)f(s^t) + \\
& + \sum_{s_{t+1} \in S} Q(s_{t+1}|s^t)B_d(s_{t+1}, s^t) + \\
& + \sum_{s_{t+1} \in S} x(s^t)Q^*(s_{t+1}|s^t)B_f(s_{t+1}, s^t) \\
= & B_d(s^t) + x(s^t)B_f(s^t) + w(s^t) + r(s^t)k(s^t) + \Pi(s^t),
\end{aligned}$$

where g is some constant that will turn out helpful later.

As we can see, the representative household uses the composite good for both consumption and investment, and its income includes labor income, capital income and profits paid by firms (equal to zero in equilibrium).

In addition, we introduce a competitive firm that by maximizing profits makes the decision how to optimally combine capital and labor to produce output. Its objective is formalized by the choice of allocation

$$D(s^t), D^*(s^t), k(s^t), l(s^t),$$

to maximize

$$\Pi(s^t) = p_d(s^t)D(s^t) + x(s^t)p_d^*(s^t)D^*(s^t) - w(s^t)l(s^t) - r(s^t)k(s^t), \quad (2.23)$$

subject to the production constraints

$$D(s^t) + D^*(s^t) = k(s^t)^\alpha (A(s^t)l(s^t))^{1-\alpha}.$$

Market clearing in the extended setup additionally requires that the supply of each type of good by the firms equals the demand:

$$\begin{aligned} d(s^t) &= D(s^t), \\ d^*(s^t) &= D^*(s^t), \\ f(s^t) &= F(s^t), \\ f^*(s^t) &= F^*(s^t), \end{aligned} \tag{2.24}$$

and that the labor market clears:

$$\begin{aligned} l(s^t) &= 1, \\ l^*(s^t) &= 1. \end{aligned} \tag{2.25}$$

Remark 45 *The particular parameterization of the production function using Cobb-Douglas utility function is justified by the fact that share of payments to labor in output is roughly constant in the data (one of the Kaldor's growth facts)—suggesting a unit elasticity between labor and capital in the aggregate production function.*

Exercise 46 *Demonstrate that the model with capital laid out above is equivalent to a combination of our prototype endowment model with an additional problem solved by a representative international firm given by:*

$$\max \sum_t \sum_{s^t} \beta^t \hat{Q}(s^t) [p_d y(s^t) + p_f(s^t) y^*(s^t)]$$

subject to

$$\begin{aligned}
y(s^t) &= A(s^t)^{1-\alpha} k(s^t)^\alpha - I_d(s^t) - I_d^*(s^t) \\
y^*(s^t) &= A^*(s^t)^{1-\alpha} k^*(s^t)^\alpha - I_f(s^t) - I_f^*(s^t) \\
i(s^t) &= G(I_d(s^t), I_f(s^t)), \\
i^*(s^t) &= G(I_f^*(s^t), I_d^*(s^t)), \\
gk(s^{t+1}) &= (1 - \delta)k(s^t) + i(s^t), \\
gk^*(s^{t+1}) &= (1 - \delta)k^*(s^t) + i^*(s^t),
\end{aligned}$$

where g is some constant (can be 1), and $\hat{Q}(s^t)$ is defined recursively as $\hat{Q}(s^t) \equiv \hat{Q}(s^{t-1})Q(s_t|s^t)$. *HINT: Use the fact that first order conditions are necessary and sufficient, equilibrium exists and is unique.*

Labor/Leisure Choice

In the baseline model, households inelastically supply all the labor. To incorporate labor/leisure choice, we will use the following utility function:

$$u(c, l) = \frac{(c^\eta(1-l)^{1-\eta})^{1-\theta}}{1-\theta}.$$

(Note that with labor-leisure choice, the existence of balanced growth path requires technological progress to be labor augmenting, i.e. $y = k^\alpha(Al)^{1-\alpha}$.)

Remark 47 *The choice of this utility function is justified by the fact that per capita leisure in the post-war period is roughly constant. At the same time, real wages have been increasing steadily. Taken together, these two observations suggest a unit elasticity between consumption and leisure, which is assumed above.*

To characterize the equilibrium, in the set of our first order conditions, we would need to additionally include the following equations: (i) Euler equation for capital:

$$u_c(s^t)g = \beta E_{s^t}[u_c(s^{t+1})((1-\delta) + r(s^{t+1}))], \quad (2.26)$$

(ii) labor leisure choice condition:

$$\frac{u_l(s^t)}{u_c(s^t)} = -w(s^t), \quad (2.27)$$

and (iv) factor prices:

$$\begin{aligned} r(s^t) &= \alpha p_d(s^t) k(s^t)^{\alpha-1} (A(s^t) l(s^t))^{1-\alpha}, \\ w(s^t) &= (1-\alpha) A(s^t) p_d(s^t) k(s^t)^\alpha (A(s^t) l(s^t))^{-\alpha}, \end{aligned}$$

where analogous conditions for the foreign country apply.

Final remarks on the complete asset market assumption? At this point, you may wonder to what extent it makes sense to assume completeness of asset market? To what extent the results discussed above carry through to economies that restrict the asset span?

Surprisingly, the answer is that the assumption completeness of asset markets is almost without loss of generality in this particular environment. Specifically, it turns out that the allocation in an analogous frictionless model with incomplete markets²⁷ exactly coincides with the allocation of the complete asset market economy. Concluding, if one believes that the allocation in the data is far away from a complete asset market allocation, one should rethink the assumptions that make this ‘equivalence result’ hold, which is obviously way more difficult.

The key thing that makes the restriction of asset space not so relevant in this class of models is that under incomplete markets, a portfolio of different types of

²⁷By incomplete markets I mean a symmetric setup in which either two state uncontingent bonds are traded, domestic and foreign, or households can hold domestic or foreign equity.

bonds can deliver almost full risk sharing. The key reason is that when, for example, the exchange rate appreciates during recessions, by simply taking a short position on the home-bond and a long position on the foreign bond, the household can hedge its consumption risk and obtain this state of the world a transfer from abroad. A similar completion of markets can be achieved more generally using a set of bonds but with a different maturity. These results have been established in the series of papers by Lucas (1982), Gourinchas and Coeurdacier (2008), Heathcote and Perri (2007), and under sticky prices, Engel and Matsumoto (2008). Typically, to get any action from incompleteness of market, the literature considers the asymmetric case of only one state uncontingent bond (denominated in, say, foreign consumption). Also, with more types of shocks, two bonds may not in general be sufficient to provided full risk sharing, and again restriction of asset span may matter. It is not obvious, however, how to depart from the complete markets in a way that is sensible and at the same time gives some action.

To setup an incomplete markets economy, we typically restrict attention to state uncontingent asset span of payoffs in the domestic numeraire at home and in the foreign numeraire abroad. This restriction imposes the following feasibility on bond trade:

$$\begin{aligned}
 B_d(s_{t+1}|s^t) &= B_d(s'_{t+1}|s^t), \\
 B_f(s_{t+1}|s^t) &= B_f(s'_{t+1}|s^t), \\
 B_d^*(s_{t+1}|s^t) &= B_d^*(s'_{t+1}|s^t), \\
 B_f^*(s_{t+1}|s^t) &= B_f^*(s'_{t+1}|s^t), \text{ all } s_{t+1}, s'_{t+1}, s^t.
 \end{aligned}$$

It can be imposed either directly on the household's problem (i.e. built into notation), or as a feasibility restriction.²⁸

²⁸Sometimes researcher restrict trade to just one bond. In such case, risk sharing can be hindered. However, this friction seems to be somewhat arbitrary.

2.5 Calibration

Our model economy is parameterized by the following 5 parameters $(\sigma, \theta, \omega, \eta, \alpha)$, and the stochastic process that governs the technology shocks: A and A^* . Since it is difficult and also pointless to explore the implications of the model for all possible combinations of the parameters, real business cycle literature calibrates the values of the parameters so that the model is broadly consistent with the facts that do not pertain to the business cycle directly, where calibrating means *to standardize as a measuring instrument*.

The basic idea underlying calibration is to exploit the fact that business cycle models also have long-run and cross-sectional predictions (first moments), and it is reasonable to require that the parameter values are such that the model is consistent with these observations (e.g. share of leisure in time endowment, or depreciation of capital etc...).²⁹

To calibrate the model, we identify the domestic country with the US, and the foreign country with an aggregate of 15 major European countries *plus* Switzerland, Japan, Canada, and Australia. Whenever possible, we assume symmetry and use the US data to calibrate a parameter, and only when necessary use the data from rest of the world.

Since there are many parameters to calibrate, it is convenient to separate our analysis to the choice of parameters that are common to the open economy and the underlying closed economy, and then consider separately the parameters specific to the open economy.

²⁹In other words, the calibration starts from the premise that the same model should be used to account for both the business cycle and the long-run observations, as the theory bundles these aspects together. In dynamic macroeconomics, the distinction between a growth model and business cycle model is artificial—this is the implication of the theory.

Parameters common to open economy and closed economy

In this section, we calibrate the values of the parameters that are common to open economy model and the underlying closed economy model. To calibrate their values, we require that the balanced growth path implications of the model are consistent with the long-run mean values in the US data. This group includes: $\alpha, \beta, \delta, \theta$.

The references for the procedure we follow below are: (1) the classic paper³⁰ by Cooley and Prescott, “Economic Growth and Business Cycle,” in Thomas Cooley, ed., “Frontiers of Business Cycle Research,” Princeton, NJ: Princeton University Press, 1995, pp.1-38, and two more recent papers by (2) Gomme, Paul & Peter Rupert (2007): “Theory, Measurement and Calibration of Macroeconomic Models”, Journal of Monetary Economics, Volume 54, Issue 2, March 2007, Pages 460-497, and (3) Gomme, Rupert, Ravikumar (2006): “Returns to Capital and the Business Cycles,” Federal Reserve Bank of Cleveland WP06-03, Feb. With respect to these papers, our analysis below will be simplified. For example, we will ignore an explicit consideration of taxes in the model, home production, or durable consumption. (It won’t change much the values of parameters, though.)

Under the assumption of symmetry, it is not difficult to see that the following standard neoclassical growth model describes the balanced growth path of our two country model (each country separately, as well as the entire world):

$$\max \sum_t \hat{\beta}^t \frac{(C_t^\eta (N_t h - L_t)^{1-\eta})^{1-\theta}}{1-\theta} \quad (2.28)$$

³⁰The book is a bit out of date, especially in terms of methods. Nevertheless, I encourage you to read this paper and the BKK 95 paper included in chapter 11.

subject to

$$\begin{aligned} C_t + I_t &= Y_t, \\ Y_t &= K_t^\alpha (\gamma^t L_t)^{1-\alpha} \\ K_{t+1} &= (1 - \delta)K_t + I_t, \end{aligned}$$

where capital letters stand for aggregate variables, γ is labor augmenting technological progress and N_t is the size of the working age population, and h is the total time endowment of non-sleeping hours per working age person (assumed 7 x 15h per day = 105h per week).

The working age population size is assumed to be equal to the product of a constant share of working age population in total population *times* the size of the total population P_t that grows at some constant rate ζ . These assumptions, up to the fluctuations in the share due to WWII baby-bust and post-war baby-boom, are consistent with the data, as illustrated in Figure 2.3.

Since the share of working age population in total population is roughly constant, we may normalize $\frac{N_t h}{P_t}$ to 1 and translate the model to per capita terms by defining the corresponding per capita variables as follows: $C_t = \zeta^t c_t$, $K_t = \zeta^t k_t$, $L_t = l h N_t = l \zeta^t$, $N_t = \zeta^t / h$, $P_t = \zeta^t$ (where $\zeta = \frac{P_{t+1}}{P_t}$ denotes population grow). Substituting out and simplifying whenever possible, we obtain

$$\max \sum_t \beta^t \frac{(c_t^\eta (1 - l_t)^{1-\eta})^{1-\theta}}{1 - \theta} \quad (2.29)$$



Figure 2.3: Share of working age population and population growth in the US.

subject to

$$\begin{aligned} c_t + i_t &= y_t, \\ y_t &= k_t^\alpha (\gamma^t l_t)^{1-\alpha} \\ \zeta k_{t+1} &= (1 - \delta)k_t + i_t, \end{aligned}$$

where $\beta \equiv \hat{\beta}\zeta$.

The first order conditions to the maximization problem stated above can be derived from the following Lagrangian

$$\mathcal{L} = \sum_t \beta^t \frac{(c_t^\eta (1 - l_t)^{1-\eta})^{1-\theta}}{1 - \theta} - \sum_t \lambda_t (c_t + \zeta k_{t+1} - (1 - \delta)k_t - k_t^\alpha (\gamma^t l_t)^{1-\alpha}).$$

The first order conditions are:

$$\begin{aligned} \beta^t \eta c_t^{\eta-1} (1 - l_t)^{1-\eta} (c_t^\eta (1 - l_t)^{1-\eta})^{-\theta} &= \lambda_t, \\ \beta^t (1 - \eta) c_t^\eta (1 - l_t)^{-\eta} (c_t^\eta (1 - l_t)^{1-\eta})^{-\theta} &= \lambda_t \gamma^t (1 - \alpha) k_t^\alpha (\gamma^t l_t)^{-\alpha}, \\ \lambda_t \zeta &= \lambda_{t+1} [(1 - \delta) + \alpha k_{t+1}^{\alpha-1} (\gamma^{t+1} l_{t+1})^{1-\alpha}], \end{aligned}$$

and can be compactly summarized by the following 3 conditions: (i) the familiar Euler equation

$$\zeta = \frac{\beta c_{t+1}^{-1} (c_{t+1}^\eta (1 - l_{t+1})^{1-\eta})^{1-\theta}}{c_t^{-1} (c_t^\eta (1 - l_t)^{1-\eta})^{1-\theta}} [(1 - \delta) + \alpha k_{t+1}^{\alpha-1} (\gamma^{t+1} l_{t+1})^{1-\alpha}], \quad (2.30)$$

(ii) labor/leisure choice condition

$$\frac{c_t}{1 - l_t} \frac{1 - \eta}{\eta} = \gamma^t (1 - \alpha) k_t^\alpha (\gamma^t l_t)^{-\alpha}, \quad (2.31)$$

and (iii) aggregate feasibility condition

$$c_t + \zeta k_{t+1} - (1 - \delta)k_t = k_t^\alpha (\gamma^t l_t)^{1-\alpha}. \quad (2.32)$$

The supporting prices (gross rental price of capital and the wage rate) can be calculated as follows:

$$\begin{aligned} r_t &= \alpha k_t^{\alpha-1} (\gamma^t l_t)^{1-\alpha}, \\ w_t &= (1 - \alpha) \gamma^t k_t^\alpha (\gamma^t l_t)^{-\alpha}. \end{aligned} \quad (2.33)$$

Balanced Growth Path Here, we solve for the balanced growth path (BGP) of this model. By definition, the balanced growth path is given by:

$$\begin{aligned} c_t &= \gamma_c^t \hat{c}, \\ k_t &= \gamma_k^t \hat{k}, \\ i_t &= \gamma_i^t \hat{i}, \\ y_t &= \gamma_y^t \hat{y}, \\ l_t &= \gamma_l^t \hat{l}, \end{aligned} \quad (2.34)$$

where γ_j denotes the growth rate of the corresponding variable j , and $\hat{\cdot}$ are the initial values of the variables.

Our task is to find the growth rates $\gamma_c, \gamma_k, \gamma_i, \gamma_y, \gamma_l$ and initial values $\hat{c}, \hat{k}, \hat{i}, \hat{y}, \hat{l}$ so that the implied balanced growth path sequence $\{c_t, k_t, i_t, y_t, l_t\}_t$ solved the planning problem stated above. The proposition below establishes that in this model all variable except stationary leisure grow at the same rate γ —a property that is broadly consistent with the growth experience of the industrial countries.

Proposition 48 *The model given by (2.29) has a unique balanced growth path (BGP)*

that satisfies

$$\gamma_c = \gamma_k = \gamma_i = \gamma_y = \gamma_l = \gamma, \gamma_l = 1,$$

and the initial values $\hat{c}, \hat{k}, \hat{i}, \hat{y}, \hat{l}$ are given by the solution to the following system: (i) Euler's equation

$$\zeta = \beta \gamma^{\eta(1-\theta)-1} [(1-\delta) + \alpha \frac{\hat{y}}{\hat{k}}], \quad (2.35)$$

(ii) labor/leisure choice condition

$$\frac{\hat{c}}{1-\hat{l}} \frac{1-\eta}{\eta} = (1-\alpha) \frac{\hat{y}}{\hat{l}}, \quad (2.36)$$

and (iii) aggregate feasibility condition

$$\hat{c} + \gamma \zeta \hat{k} - (1-\delta) \hat{k} = \hat{y}, \quad (2.37)$$

$$\hat{y} = \hat{k}^\alpha \hat{l}^{1-\alpha},$$

$$\hat{i} = \gamma \zeta \hat{k} - (1-\delta) \hat{k}.$$

The supporting equilibrium prices can be found from:

$$r = \alpha \frac{\hat{y}}{\hat{k}}, \quad (2.38)$$

$$w_t = \gamma^t (1-\alpha) \frac{\hat{y}}{\hat{l}}.$$

Proof. Since we know that the solution to the planning problem stated in (2.29) is unique up to the given value of initial capital, it is sufficient to show that the proposed balanced growth path (2.34) solves the model, and the conditions (i)-(iii) uniquely pin down the values of $\hat{c}, \hat{k}, \hat{i}, \hat{y}, \hat{l}$.

Clearly, equations (2.35)-(2.38) have been obtained by substituting out for allocation from the balanced growth path into the first order conditions given by (2.30)-(2.33). To show that they imply a unique solution for $\hat{c}, \hat{k}, \hat{i}, \hat{y}, \hat{l}$, we will solve for this

solution explicitly. To this end, observe that from the first equation, we can find the value of $\frac{\hat{y}}{\hat{k}}$. Given this value, we can divide equation (2.37) by \hat{k} to find $\frac{\hat{c}}{\hat{k}}$. Finally, we can rewrite 2.36 as

$$\frac{\hat{l}}{1-\hat{l}} \frac{1-\eta}{\eta} = (1-\alpha) \left(\frac{\hat{c}}{\hat{k}}\right)^{-1} \frac{\hat{y}}{\hat{k}},$$

and obtain the value of $\frac{\hat{l}}{1-\hat{l}}$. After solving for \hat{l} , we can recover the values $\hat{c}, \hat{k}, \hat{l}, \hat{y}$.

We conclude that the balanced growth path is uniquely pinned down by conditions (i)–(iii), and it satisfies the first order conditions to the planning problem (2.29). Since first order conditions are also necessary and sufficient, the proposition follows. ■

Exercise 49 *Establish the connection between the balanced growth path for the planning problem stated in (2.29) and the balanced growth path in our original two-country model. Specifically, include d, f, d^*, f^* in the definition of the balanced growth path, and show that if these variables also growth at rate γ , such ‘extended balanced growth’ path solves the corresponding two-country planning problem that is given by:*

$$\max \left[\sum_t \beta^t \sum_{s^t \in S^t} [\pi(s^t)u(c(s^t)) + \sum_{s^t \in S^t} \beta^t \pi(s^t)u(c^*(s^t))] \right]$$

subject to

$$\begin{aligned} c(s^t) + i(s^t) &= \xi G(d(s^t), f(s^t)), \\ c^*(s^t) + i^*(s^t) &= \xi G(f^*(s^t), d^*(s^t)) \end{aligned}$$

and

$$\begin{aligned}
d(s^t) + d^*(s^t) &= y(s^t), \\
f(s^t) + f^*(s^t) &= y^*(s^t), \\
y(s^t) &= k(s^t)^\alpha (\gamma^t l(s^t))^{1-\alpha}, \\
y^*(s^t) &= k^*(s^t)^\alpha (\gamma^t l^*(s^t))^{1-\alpha}, \\
\zeta k(s^{t+1}) &= (1 - \delta)k(s^t) + i(s^t), \\
\zeta k^*(s^{t+1}) &= (1 - \delta)k^*(s^t) + i^*(s^t), \text{ all } s^t \in S^t,
\end{aligned}$$

where $G(\cdot)$ is hom of degree 1 and ξ is some arbitrary constant. Argue that, in fact, when the value of ξ is properly chosen (say what it must be), on the balanced growth path this model boils down to a simple neoclassical growth model.

Our next task is to use data to calibrate the parameters of the closed economy model. The data comes from the Economic Report of the President (available online),³¹ with the source tables denoted by B-XX. Economic Report of the President follows the layout of NIPA tables, but for our purposes it is tabulated more conveniently than NIPA. On my website you will find an Excel file with the data and the calibration discussed below.

Calibrating α In the first step, we calibrate the value of α . According to the model, α is the share of payments to labor:

$$\frac{w_t L_t}{Y_t} = \frac{w_t l_t N_t}{y_t N_t} = \frac{w_t l_t}{y_t} = 1 - \alpha,$$

To calculate the above share, we need to have total factor payments to domestic factors, and payments to labor. To obtain these values we look at the GDI NIPA

³¹See <http://www.gpoaccess.gov/eop/tables08.html>.

| Line | | 2006 | 2006 | 2006 | 2006 | 2007 | 2007 | 2007 | 2007 | 2008 | 2008 | 2008 |
|------|---|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|----------------|
| | | I | II | III | IV | I | II | III | IV | I | II | III |
| 1 | Gross domestic income | 13,114.3 | 13,265.8 | 13,420.4 | 13,565.0 | 13,699.3 | 13,881.0 | 13,958.4 | 14,017.4 | 14,087.4 | 14,196.1 | ... |
| 2 | Compensation of employees, paid | 7,324.6 | 7,370.7 | 7,448.5 | 7,618.0 | 7,715.6 | 7,767.2 | 7,846.4 | 7,948.3 | 8,017.0 | 8,078.9 | 8,142.8 |
| 3 | Wage and salary accruals | 5,933.0 | 5,972.7 | 6,040.7 | 6,194.1 | 6,275.6 | 6,317.7 | 6,384.8 | 6,472.8 | 6,525.2 | 6,575.8 | 6,630.4 |
| 4 | Disbursements | 5,953.0 | 5,975.7 | 6,040.7 | 6,169.1 | 6,300.6 | 6,317.7 | 6,384.8 | 6,472.8 | 6,525.2 | 6,575.8 | 6,630.4 |
| 5 | To persons | 5,943.5 | 5,963.3 | 6,031.3 | 6,159.3 | 6,291.0 | 6,307.7 | 6,374.8 | 6,462.5 | 6,515.0 | 6,565.6 | ... |
| 6 | To the rest of the world | 9.4 | 9.4 | 9.4 | 9.7 | 9.6 | 10.0 | 10.1 | 10.3 | 10.2 | 10.2 | ... |
| 7 | Wage accruals less disbursements | -20.0 | 0.0 | 0.0 | 25.0 | -25.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 8 | Supplements to wages and salaries | 1,391.6 | 1,398.0 | 1,407.8 | 1,423.9 | 1,440.0 | 1,449.4 | 1,461.6 | 1,475.5 | 1,491.7 | 1,503.1 | 1,512.4 |
| 9 | Taxes on production and imports | 962.7 | 973.6 | 980.1 | 988.3 | 1,002.7 | 1,012.3 | 1,019.2 | 1,027.7 | 1,025.8 | 1,039.4 | 1,038.0 |
| 10 | Less: Subsidies ¹ | 54.2 | 49.8 | 48.2 | 46.8 | 47.5 | 55.9 | 53.5 | 52.3 | 50.6 | 50.8 | 50.4 |
| 11 | Net operating surplus | 3,296.5 | 3,358.8 | 3,401.6 | 3,343.4 | 3,344.2 | 3,450.3 | 3,414.4 | 3,335.2 | 3,317.4 | 3,325.5 | ... |
| 12 | Private enterprises | 3,306.4 | 3,367.1 | 3,410.8 | 3,352.6 | 3,355.1 | 3,458.8 | 3,419.9 | 3,341.8 | 3,324.5 | 3,333.3 | ... |
| 13 | Net interest and miscellaneous payments, domestic industries | 780.0 | 807.7 | 817.6 | 850.0 | 864.8 | 897.0 | 900.1 | 936.7 | 915.4 | 935.8 | ... |
| 14 | Business current transfer payments (net) | 85.1 | 83.5 | 86.0 | 86.8 | 98.3 | 97.4 | 102.2 | 103.1 | 103.2 | 102.1 | 92.8 |
| 15 | Proprietors' income with inventory valuation and capital consumption adjustments | 1,004.7 | 1,018.3 | 1,013.4 | 1,022.4 | 1,037.2 | 1,050.2 | 1,063.8 | 1,073.8 | 1,071.7 | 1,076.9 | 1,080.0 |
| 16 | Rental income of persons with capital consumption adjustment | 52.8 | 45.6 | 40.4 | 38.2 | 35.1 | 44.6 | 41.8 | 38.6 | 39.1 | 58.6 | 64.3 |
| 17 | Corporate profits with inventory valuation and capital consumption adjustments, domestic industries | 1,383.7 | 1,412.0 | 1,453.3 | 1,355.1 | 1,319.7 | 1,369.7 | 1,311.9 | 1,189.7 | 1,195.1 | 1,159.8 | ... |
| 18 | Taxes on corporate income | 453.8 | 474.8 | 487.2 | 459.8 | 448.5 | 468.5 | 451.1 | 433.5 | 402.9 | 406.8 | ... |
| 19 | Profits after tax with inventory valuation and capital consumption adjustments | 929.9 | 937.1 | 966.0 | 895.4 | 871.2 | 901.1 | 860.8 | 756.3 | 792.1 | 753.0 | ... |
| 20 | Net dividends | 548.2 | 583.1 | 642.9 | 740.9 | 653.8 | 661.7 | 662.2 | 706.6 | 654.9 | 681.6 | ... |
| 21 | Undistributed corporate profits with inventory valuation and capital consumption adjustments | 381.6 | 354.0 | 323.1 | 154.5 | 217.5 | 239.4 | 198.6 | 49.7 | 137.2 | 71.4 | ... |
| 22 | Current surplus of government enterprises ¹ | -7.8 | -8.3 | -9.1 | -9.2 | -10.8 | -8.5 | -5.5 | -6.7 | -7.1 | -7.7 | -8.0 |
| 23 | Consumption of fixed capital | 1,582.7 | 1,612.5 | 1,638.3 | 1,662.2 | 1,684.3 | 1,707.0 | 1,731.9 | 1,758.6 | 1,778.0 | 1,803.1 | 1,900.2 |
| 24 | Private | 1,323.1 | 1,348.8 | 1,367.8 | 1,386.2 | 1,402.1 | 1,420.0 | 1,440.1 | 1,462.5 | 1,477.5 | 1,497.4 | 1,588.0 |
| 25 | Government | 259.5 | 265.8 | 270.5 | 275.9 | 282.2 | 287.0 | 291.8 | 296.3 | 300.5 | 305.7 | 312.2 |
| 26 | Addendum: | | | | | | | | | | | |
| 26 | Statistical discrepancy | -154.6 | -131.7 | -170.8 | -194.9 | -188.4 | -143.4 | -7.8 | 13.9 | 63.4 | 98.4 | ... |

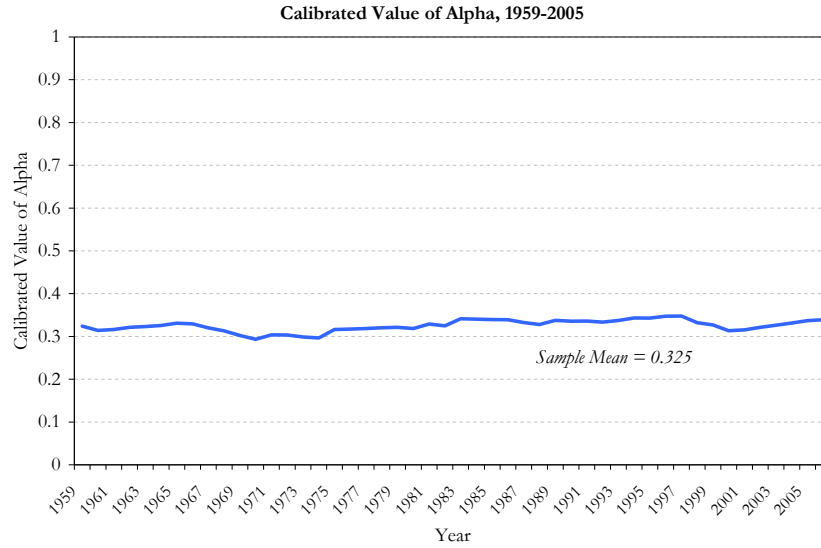
Figure 2.5: GDI (=GDP + statistica discrepancy) broken down by income type.

item 15 (proprietor's income). To deal with this problem, we assume, in consistency with our production function, that $1 - \alpha$ fraction of proprietor's income is labor income. We also subtract excise and sales taxes from GDP to obtain the 'pure' total factor payments. Our adjustments give:

- Total factor payments to capital and labor = (GDP Y_t B-1) - (taxes on production and imports *less* subsidies B-27).
- Total payments to labor = (compensation of employees B28)+(1 - α fraction of the proprietor's income B28).

The value of α can now be obtained from the following calculation

$$1 - \alpha = \frac{\text{Compensation of employees} + (1 - \alpha)\text{Proprietor's income}}{\text{GDP} - \text{Taxes on production and imports less subsidies}},$$

Figure 2.6: Calibrated values of α , 1959-2005.

which gives³⁴

$$\alpha = 1 - \frac{\text{Compensation of employees}}{\text{GDP} - \text{Taxes (..)} - \text{Proprietor's income}}.$$

The calibrated value of α for the period 1959-2005 is about 1/3, as expected. This value is slightly less than the one obtained by Cooley and Prescott. The difference is that Cooley and Prescott, in consistency with the model, included durable consumption goods in their measure of broadly defined capital. To keep things simple, we omitted this consideration. Figure 2.6 illustrates the time-series of the calibrated value for each year. As required, the parameter is approximately constant over time.

Calibrating δ To calibrate the depreciation rate of capital, we use the information on investment in capital and consumption of fixed capital (i.e. the estimate of depreciation of capital by NIPA). In this respect, we obtain

³⁴Prescott in his measure of capital included consumer durables, his α was a bit higher. It is an issue how to deal with consumer durables, certainly one way is to lump it all into capital and assume consumption is consumption of non-durables + services from durables.

- Gross investment $\equiv K_{t+1} - (1 - \delta)K_t = (\text{gross private investment B-1}) + (\text{gross government investment B-20}),$
- Depreciation $\equiv \delta K_t = (\text{consumption of fixed capital B-26}),$ and thus
- Net investment $\equiv K_{t+1} - K_t = [K_{t+1} - (1 - \delta)K_t] - \delta K_t.$

Having these time-series, we use the fact that on the balance growth path in our model, we have

$$\frac{(K_{t+1} - K_t)}{Y_t} = \frac{(k_{t+1} \frac{P_{t+1}}{P_t} - k_t) P_t}{y_t P_t} = (\gamma\zeta - 1) \frac{\hat{k}}{\hat{y}},$$

which implies:

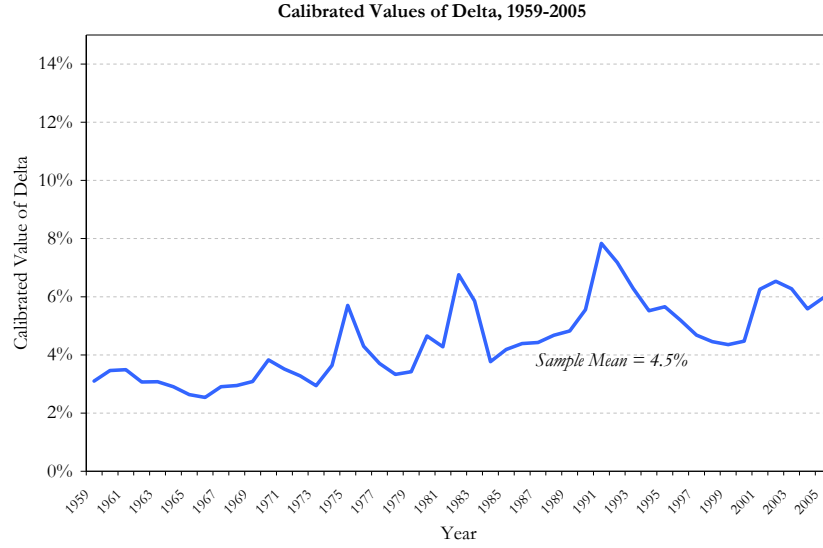
$$\frac{\hat{y}}{\hat{k}} = \frac{(\gamma\zeta - 1)Y_t}{K_{t+1} - K_t}, \quad (2.39)$$

Having the above ratio, we can calculate δ from the following identity (K_t, Y_t both grow at the same rate $\gamma\zeta$):

$$\delta \equiv \frac{\delta K_t}{Y_t} \frac{Y_t}{K_t} = \frac{\delta K_t}{Y_t} \frac{\hat{y} \gamma^t P_t}{\hat{k} \gamma^t P_t} = \frac{\delta K_t}{Y_t} \frac{\hat{y}}{\hat{k}}.$$

To calculate δ , we have used micro-level information about depreciation of capital from NIPA. The whole calculation was just correcting for the fact that part of the investment must be used to augment capital in consistency with the BGP—rest followed from the aggregate estimates of capital depreciation by BEA.

The average value of δ for the period 1959-2005 is about 4.5% per annum (1.14% per quarter). Figure 2.7 illustrates the underlying time-series of the calibrated values for each year. As we can see, it is not as nice and stationary as α , but the secular trend is small. It also fluctuates quite a bit due to the high volatility of investment over the business cycle—which is nothing to be worried about. We know that we are not looking at the ‘pure’ balanced growth path in the data. The secular trend is

Figure 2.7: Calibrated values of δ , 1959-2006.

more problematic. It suggests that the economy may not be exactly on the balanced growth path the way our model looks at it.

Calibrating β On the balanced growth path, Euler's equation implies

$$\zeta = \beta \gamma^{-1} \gamma^{a(1-\theta)-1} [(1 - \delta) + \alpha k_{t+1}^{\alpha-1} (\gamma^{t+1} l_{t+1})^{1-\alpha}],$$

and thus

$$\beta = \frac{\zeta \gamma^{\eta(\theta-1)+1}}{\alpha \frac{\hat{y}}{\hat{k}} + 1 - \delta}. \quad (2.40)$$

Since we lack the values of θ and η , and can not compute β yet. Prescott chose $\theta = 1$ on the basis that such value is roughly consistent with the spread between the real rates of return in countries with the highest and lowest consumption growth. In such case, β can be calculated right away because η cancels out. Later, more evidence become available to pin down θ . Experiments of risk aversion point to values between

1-3 (see info in Mehra and Prescott (1985)), and time series analysis by Eichenbaum, Hansen and Singleton (1988) suggests values centered around 2, that we should also consider. (The value of $\theta = 2$ is widely used in the literature; Prescott uses $\theta = 1$.)

When $\theta = 2$, we need to pin down η first. The value of η can be obtained by calculating the ratio of work in total time endowment, which gives \hat{l} . We calculate it by evaluating the ratio of the total number of workers to size of total working age population, and multiply it by the ratio of the average number of hours worked per average worker to the total endowment of non-sleep hours; assumed to be 105h per week (15h of non-sleeping time per day).³⁵ The obtained this way value of \hat{l} oscillates at around 1/3, and implies the value of $\frac{1-\hat{l}}{\hat{l}}$ equal to about 2. (The data on hours comes from CPS census³⁶, and the number is consistent with other micro-level studies pointing to slightly lower number of 30% (e.g. Juster and Stafford (1991), “The Allocation of Time ...”. *Journal of Economic Literature*, 29:471:522)

Having calculated the share of market activities in total time endowment, from the first order condition on labor/leisure choice, we thus derive:

$$\frac{\hat{c}}{1-\hat{l}} \frac{1-\eta}{\eta} = (1-\alpha) \frac{\hat{y}}{\hat{l}}$$

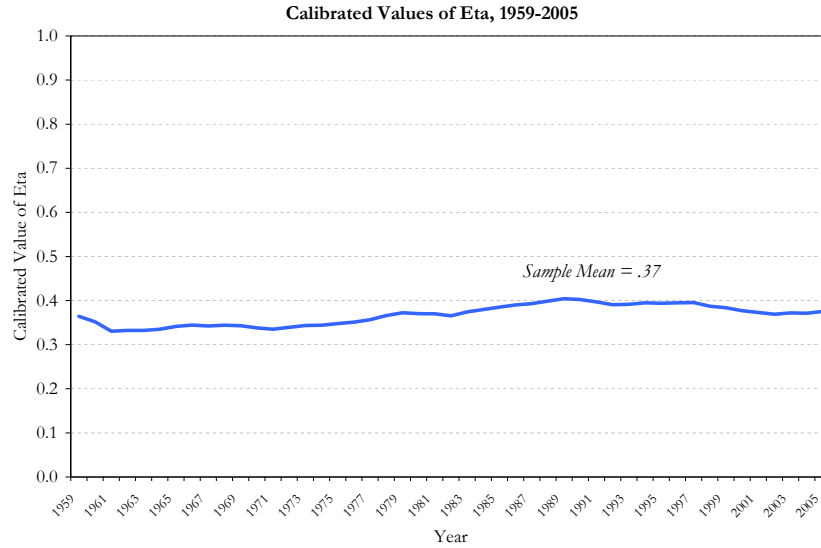
and obtain the value of η from

$$\begin{aligned} \frac{1-\eta}{\eta} \frac{\hat{l}}{1-\hat{l}} &= (1-\alpha) \frac{\hat{y}}{\hat{c}} = (1-\alpha) \frac{Y_t}{C_t}, \\ \frac{1-\eta}{\eta} &= \frac{1-\hat{l}}{\hat{l}} (1-\alpha) \frac{Y_t}{C_t}, \\ \eta &= \frac{1}{\frac{1-\hat{l}}{\hat{l}} (1-\alpha) \frac{Y_t}{C_t} + 1}. \end{aligned}$$

(To calculate $\frac{Y_t}{C_t}$, we correct for the impact of the large negative NX, and calculate

³⁵See formulas in the Excel file on my website for more detail.

³⁶Source: Cociuba, Ueberfeld and Prescott (2007).

Figure 2.8: Calibrated values of η , 1959-2006.

$\frac{Y_t}{C_t}$ by evaluating: $\frac{Y_t}{C_t} = \frac{Y_t + NX}{Y_t + NX - [K_{t+1} - (1 - \delta)K_t]}$.³⁷ The correction does not make much difference (see Excel file posted online.)

The average value of η calculated this way is .37, and having η , we can calculate β from the formula:

$$\beta = \gamma^{\eta(\theta-1)+1} [(1 - \delta) + \alpha \frac{\hat{y}}{\hat{k}}]^{-1}.$$

Figures 2.8-2.9 illustrate the time-series for the calibrated values of η and β for each year. As required, both are roughly stationary.

Last but not least, we have to translate our model to quarterly frequency (our estimates are annual). In order to do this, we calculate implied quarterly depreciation rate and discount rate from the formula for compounded change as follows: $\delta_q = 1 - (1 - \delta_a)^{\frac{1}{4}}$, and $\beta_q = \beta_a^{\frac{1}{4}}$. The summary of the obtained this way parameter values for a quarterly model is given in Table 2.4.

³⁷In consistency with theory, we are treating here NX as part of domestic output. This is the total that in our model is split by households into consumption and investment.

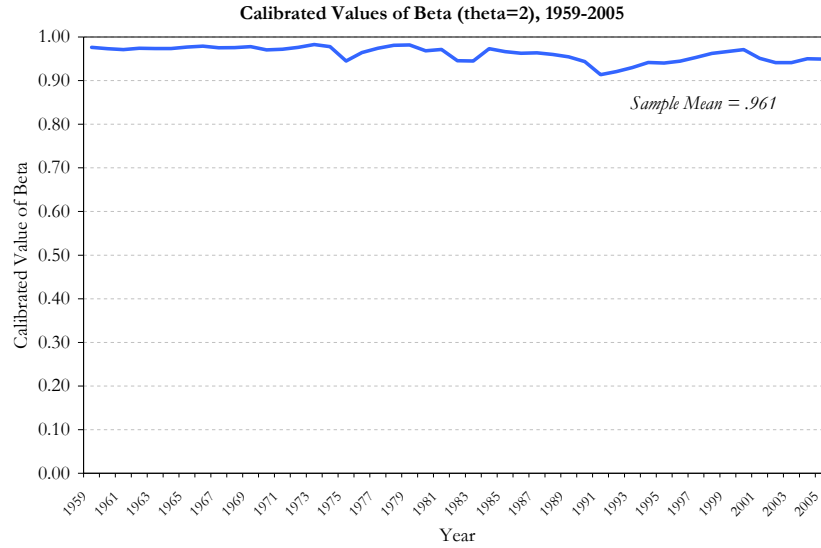


Figure 2.9: Calibrated values of β for $\theta = 2$, 1959-2006.

Pre-tax real net return on capital Given the values of the calibrated parameters, using Euler's equation, we next calculate the net return on capital (note that this return is risky when there is business cycle):³⁸

$$r - \delta = \alpha \frac{\hat{y}}{\hat{k}} - \delta = \frac{\gamma^{\eta(\theta-1)+1}}{\beta} - 1 + \delta - \delta.$$

In our model, it is on average equal to 8.2%. This is not too bad, but slightly too high comparing to the pre-tax real return calculated directly from NIPA data of about 6% (see the paper by Ravikumar et al.).

Exercise 50 Using data from National Accounts available at www.sourceOECD.org and data available from <http://www.statistik.at/web-en/statistics/index.html>, calibrate the parameters $\alpha, \delta, \eta, \beta$ for Austria (AUT). How do these values compare with the values listed above for the US?

³⁸To use the analogy to our earlier analysis, return r must satisfy along the balance growth path the equation $1 = \beta \frac{u'(c_{t+1})}{u'(c_t)}(1 + r)$.

Table 2.4: Parameter values for the quarterly model.

| Parameter | Parameter value | |
|-----------|-----------------|--------------|
| | $\theta = 2$ | $\theta = 1$ |
| γ | 1.021 | 1.021 |
| ζ | 1.011 | 1.011 |
| α | 0.325 | 0.325 |
| δ | 0.014 | 0.014 |
| η | 0.367 | 0.367 |
| β | 0.988 | 0.99 |

Notes: Based on mean values estimated from annual US data (1959-2006) and translated to quarterly frequency.

Calibrating parameters specific to the open economy

Our open economy model adds two additional parameters that we need to discipline. These parameters are the elasticity of substitution between domestic and foreign goods σ , and the home-bias parameter ω .

The most reasonable thing to do would be to adopt the value of σ consistent with long-run oriented trade literature. In the previous chapter, we have argued that trade literature suggests values centered around 5-15. This value followed, for example, from the cross-sectional estimate of θ in Eaton and Kortum (2002) (recall $\theta - 1$ is isomorphic to σ), or the studies based on the impact of tariff reductions on trade (e.g. Head and Ries (2001): “*Increasing Returns (...)*,” American Economic Review, 91(4), pp. 858-976). Also, the analysis by Anderson and Wincoop (2003) pointed us to such values.

Given the value of σ , finding ω is an easy task. It is sufficient to require that the theory is consistent with the observed imports to GDP ratio. In the symmetric steady state, the two country model implies

$$f_t = \left(\frac{\omega}{1 - \omega}\right)^{-\sigma} d_t,$$

which we can use to back out ω . To see this, define the import share as

$$is = \frac{f_t}{d_t + f_t},$$

and note that import share can be measured from the data by imports/GDP ratio.

Next, rewrite the above condition to calculate:

$$\omega = \frac{\left(\frac{1-is}{is}\right)^{\frac{1}{\sigma}}}{\left(\frac{1-is}{is}\right)^{\frac{1}{\sigma}} + 1}. \quad (2.41)$$

As long as we have σ , we have ω .

Long-run versus short-run elasticity puzzle Since firm evidence on a reasonable value for σ came later, early business cycle literature used an alternative strategy to pin down this parameter. According to the theory, both approaches should give the same answer. However, the puzzle is that they do not, which has been later labeled *long-run versus short-run elasticity puzzle*.

The alternative strategy of measuring elasticity σ is motivated by the fact that in the business cycle models, the demand for domestic and foreign good is modeled by a CES aggregator. In such case, it is straightforward to show that the import ratio is tied to the relative price of domestic and imported goods by (see equations 2.13, (iii)-(iv))

$$\log \frac{f_t}{d_t} = \sigma \log \frac{p_{d,t}}{p_{f,t}} + \log \frac{\omega_t}{1 - \omega_t}. \quad (2.42)$$

(To be more general, we are allowing here for ω to be time-varying.)

Under normal conditions (i.e., when the supply curve is an upward-sloping function of the price and the supply shocks are not correlated with the ω_t -demand shocks), we should expect the correlation between $\log \frac{\omega_t}{1-\omega_t}$ and $\log \frac{p_{d,t}}{p_{f,t}}$ to be positive. But then,

the *volatility ratio* defined by

$$VR \equiv std(\log \frac{f_t}{d_t}) / std(\log \frac{p_{d,t}}{p_{f,t}}) \quad (2.43)$$

places an upper bound on the value of the intrinsic price elasticity of trade flows σ , as implied by the following evaluation of (2.42) and variance decomposition:

$$\sigma = std(\log \frac{f_t}{d_t}) / std(\log \frac{p_{d,t}}{p_{f,t}} + \frac{1}{\sigma} \log \frac{\omega_t}{1 - \omega_t}) \leq \quad (2.44)$$

$$\leq std(\log \frac{f_t}{d_t}) / std(\log \frac{p_{d,t}}{p_{f,t}}) = VR. \quad (2.45)$$

In particular, in the Armington model with ω assumed constant, the volatility ratio is exactly equal to the elasticity of substitution σ .

This is the measurement of short-run elasticity that I used with Jaromir in our customer capital paper. It avoids the use of time-series regressions. Regression in this context may require to specify a model with error correction. This gives the upper bound of the regression coefficient. In most applications the upper bound is enough for the analysis at hand.

The computed values of the volatility ratio for the data are shown in Table 2.5. As we can see, these values are very low and grossly at odds with the value of the parameter σ implied by the trade literature.³⁹ At business cycle frequencies, the median value of the volatility ratio is as low as 0.7 for both HP-filtered and linearly detrended data.

Because of this discrepancy, in our quantitative analysis of the model, we will report the results for both low and high values of σ . Namely, we will consider the values based on the estimates from Head and Ries of $\sigma = 7.9$, as well as the value $\sigma = .73$ consistent with low value of the volatility ratio in the data.⁴⁰ As we will later

³⁹Similar results, using different method, are obtained by Wilson (2001) or Reinert et al. (1992).

⁴⁰We choose the value 1 instead of .7 because we can solve this Cobb-Douglas case analytically when $\theta = 1$, and $\eta = 1$. It will be helpful later to understand the intuition.

Table 2.5: Volatility Ratio in a Cross-Section of Countries.

| Country | Detrending method | |
|-----------------|-------------------|---------------------|
| | HP-1600 | Linear ^a |
| Australia | 0.94 | 0.93 |
| Belgium | 0.57 | 0.50 |
| Canada | 1.27 | 0.64 |
| France | 0.54 | 0.73 |
| Germany | 0.90 | 1.16 |
| Italy | 0.69 | 0.46 |
| Japan | 0.60 | 0.43 |
| Netherlands | 0.44 | 0.72 |
| Switzerland | 0.71 | 1.16 |
| Sweden | 0.95 | 0.95 |
| UK | 0.65 | 0.61 |
| US ^b | 1.23 | 1.02 |
| MEDIAN | 0.71 | 0.73 |

Notes: Based on quarterly time-series, 1980 : 1 – 2000 : 1. Data sources are listed in Drozd and Nosal (2008).

^aLinear trend subtracted from logged time series.

^bFor the entire postwar period (1959 : 3 – 2004 : 2) this ratio in U.S. is 0.88.

see, the model is going to perform much better in the latter case.

Estimating the forcing process The prediction of the model crucially depends on the properties of the stochastic process for output and technology A . Below, we first back out the process for output that we use in conjunction with the endowment economy model, and then we back out the process for the Solow residuals that we use in conjunction with the extended model. The details are available from the Excel file posted on my website.

Endowment shocks To back out endowment process from the data, we must calculate the aggregate output of the rest of the world. The difficulty is that the real GDP is expressed in units that can not be readily compared (local currency units).

To adjust the units of real GDP, we use a PPP adjusted GDP for a chosen year, say year 2000, and normalize real GDP of each country so that it is 1 in year 2000.

Next, we multiply each country series by the value of PPP adjusted GDP in year 2000 pulled out from the Penn World Tables. Note that we can follow a similar procedure to aggregate virtually any aggregate time-series. For example, if we want to aggregate investment, we must know the share of investment in GDP in year 2000 in each country, and after normalizing investment series in each country so that it is 1 in year 2000, to obtain series that we can add up, it suffices to multiply each individual series by the PPP adjusted GDP in year 2000 and the share of investment in GDP for the year 2000. (The choice of the baseline year, here year 2000, matters in general, but does not change the results drastically.)

From the data, we need the real GDP series and population series for countries that we define as the rest of the world, and the US. Then, we divide output by population to translate it to per capita terms. The source of the aggregate data is OECD (www.sourceOECD.org) and Penn World Tables for population (smoothly extrapolated from the growth rate of population). Using this data, we estimate the process of the form:

$$\begin{aligned}\log y(s^{t+1}) &= \xi_1 + \gamma t + \log \hat{y}(s^{t+1}) \\ \log y^*(s^{t+1}) &= \xi_2 + \gamma t + \log \hat{y}^*(s^{t+1}) \\ \log \hat{y}(s^{t+1}) &= \rho \log \hat{y}(s^t) + \phi \log y^*(s^t) + \varepsilon(s^{t+1}), \\ \log y(s^{t+1}) &= \rho \log y^*(s^t) + \phi \log \hat{y}(s^t) + \varepsilon^*(s^{t+1}),\end{aligned}$$

where γ is a common growth parameter, ρ is persistence parameter, ϕ is spillover parameters, and $\varepsilon, \varepsilon^*$ are i.i.d. normally distributed random variables with a symmetric variance-covariance matrix Σ .

To estimate this system we proceed as follows. We estimate the model using SUR (seemingly unrelated regression) method with symmetry restrictions. We check if the spillover parameter ϕ is statistically different from zero, and if not, we reestimate the

model with a restriction $\phi = 0$. Using the estimated parameters, we then back out the regression error vector $(\hat{\varepsilon}, \hat{\varepsilon}^*)$, and fit a two-dimensional Normal distribution with a symmetry restriction imposed. The results of the estimation give:

$$\begin{aligned}\rho &= .95 \text{ (0.02)} \\ \phi &= 0.0, \\ \Sigma &= \begin{bmatrix} 3.56 & 1.17 \\ 1.17 & 3.56 \end{bmatrix} \times 10^{-5}.\end{aligned}$$

The variance-covariance matrix implies that correlation of residuals $\varepsilon, \varepsilon^*$ is .33, and the standard deviation is about .006. The 5% confidence interval on ρ is [.90, .99].

Technology shocks To recover Solow residuals, we use the following formula (based on log of $Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$):

$$\log A_t = \frac{\log Y_t - \alpha \log K_t + (1 - \alpha) \log L_t}{1 - \alpha},$$

where Y_t is real GDP, K_t is total stock of capital backed out from real investment series using perpetual inventory method⁴¹, and L_t are total hours worked for US and total civilian employment for the rest of the world. Our underlying data is quarterly and pertains to the time period 1980-2005.

Given series for A_t for US and A_t^* for the rest of the world, following the same

⁴¹The basic idea behind perpetual inventory method is to cumulate capital using equation: $K_{t+1} = (1 - \delta_t)K_t + I_t$, where I_t are the series for real gross investment, and δ_t is our calibrated value for each year (or a constant mean value δ). The initial capital K_0 is obtained from an educated guess that requires that the average growth rate of capital over the first 10 years is the same as in the initial year, i.e.

$$\frac{K_1 - K_0}{K_0} = \frac{1}{10} \sum_{i=1, \dots, 10} \frac{K_{i+1} - K_i}{K_i}.$$

procedure as with output, we estimate the process:

$$\begin{aligned}
 \log A(s^{t+1}) &= \xi_1 + \gamma t + \log \hat{A}(s^{t+1}), \\
 \log A^*(s^{t+1}) &= \xi_2 + \gamma t + \log \hat{A}^*(s^{t+1}), \\
 \log \hat{A}(s^{t+1}) &= \rho \log \hat{A}(s^t) + \phi \log \hat{A}^*(s^t) + \varepsilon(s^{t+1}), \\
 \log \hat{A}^*(s^{t+1}) &= \rho \log \hat{A}^*(s^t) + \phi \log \hat{A}(s^t) + \varepsilon^*(s^{t+1}),
 \end{aligned} \tag{2.46}$$

where as before γ is a common growth parameter, ρ is persistence parameter, ϕ is spillover parameters, and $\varepsilon, \varepsilon^*$ are *i.i.d.* normally distributed symmetric innovations with a variance-covariance matrix Σ .

The results are:

$$\begin{aligned}
 \rho &= .91 \text{ (0.025)} \\
 \phi &= 0.0, \\
 \Sigma &= \begin{bmatrix} 5.12 & 1.20 \\ 1.20 & 5.12 \end{bmatrix} \times 10^{-5},
 \end{aligned}$$

where the variance-covariance matrix implies that correlation of residuals $\varepsilon, \varepsilon^*$ is .23, and standard deviation about .0071. The 5% confidence interval on ρ is [.86, .96]).

Lastly, as a consistency check, having the series for capital, we can look up K/Y ratio in the data. According to the model, it should exhibit no trend. In Figure 2.10, we illustrate the actual time series for K/Y in the data (with initial point set equal to 3.5) over the sample period 1980-2005. As we can see, it exhibits no trend. You will find these time-series in the Excel file posted online.

Solving the Model

Having the parameter values, we next solve the model numerically. This will allow us to generate artificial data, and compare it to the actual data. Our comparison

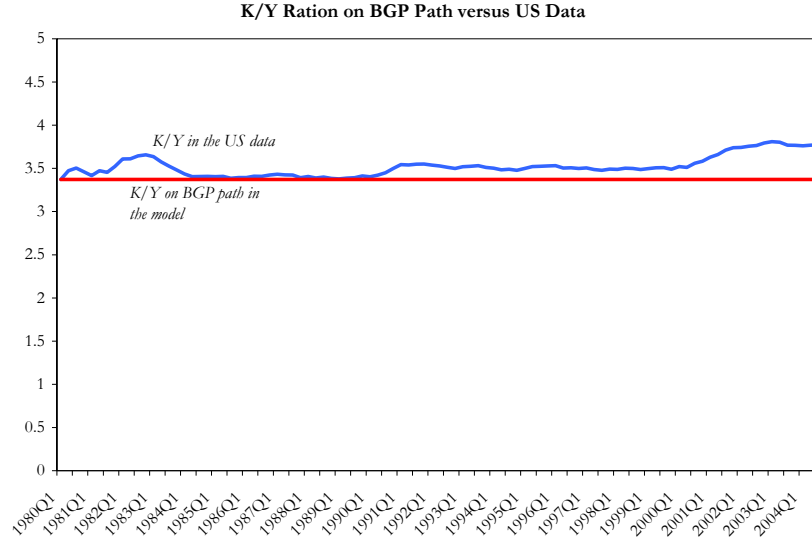


Figure 2.10: Comparison of K/Y ratio between model (BGP path) and US data (initial value matched by construction).

will focus on the basic statistics measuring volatility and comovement of time-series. These are not the only characteristics of time-series that can be compared, but they are first order statistics, and we will focus on them.

Our model is given by the following planning problem:

$$\max \left[\sum_t \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c(s^t)) + \sum_t \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c^*(s^t)) \right]$$

subject to

$$\begin{aligned} c(s^t) + i(s^t) &= G(d(s^t), f(s^t)), \\ c^*(s^t) + i^*(s^t) &= G(f^*(s^t), d^*(s^t)) \end{aligned}$$

and

$$\begin{aligned}
d(s^t) + d^*(s^t) &= y(s^t), \\
f(s^t) + f^*(s^t) &= y^*(s^t), \\
y(s^t) &= k(s^t)^\alpha (A(s^t)l(s^t))^{1-\alpha}, \\
y^*(s^t) &= k^*(s^t)^\alpha (A^*(s^t)l^*(s^t))^{1-\alpha}, \\
\zeta k(s^{t+1}) &= (1 - \delta)k(s^t) + i(s^t), \\
\zeta k^*(s^{t+1}) &= (1 - \delta)k^*(s^t) + i^*(s^t), \text{ all } s^t \in S^t,
\end{aligned}$$

where the process for A and A^* is given by (2.46).

To solve this model, we redefine all the variable by dividing them by the corresponding growth rate of the economy (by analogy for foreign country variables):

$$\begin{aligned}
\hat{c}_t &\equiv \frac{c_t}{\gamma^t}, \\
\hat{k}_t &\equiv \frac{k_t}{\gamma^t}, \\
\hat{i}_t &\equiv \frac{i_t}{\gamma^t}, \\
\hat{y}_t &\equiv \frac{y_t}{\gamma^t}, \\
\hat{l}_t &\equiv l_t, \\
\hat{d}_t &\equiv \frac{y_t}{\gamma^t}, \\
\hat{f}_t &\equiv \frac{y_t}{\gamma^t},
\end{aligned}$$

and by plugging in these values, we rewrite the original problem in a stationary form:

$$\max \left[\sum_t (\beta \gamma^{\eta(1-\theta)})^t \left[\sum_{s^t \in S^t} \pi(s^t) u(\hat{c}(s^t)) + \sum_{s^t \in S^t} \pi(s^t) u(\hat{c}^*(s^t)) \right] \right]$$

subject to

$$\begin{aligned}\hat{c}(s^t) + \hat{i}(s^t) &= G(\hat{d}(s^t), \hat{f}(s^t)), \\ \hat{c}^*(s^t) + \hat{i}^*(s^t) &= G(\hat{f}^*(s^t), \hat{d}^*(s^t))\end{aligned}$$

and

$$\begin{aligned}\hat{d}(s^t) + \hat{d}^*(s^t) &= \hat{y}(s^t), \\ \hat{f}(s^t) + f^*(s^t) &= \hat{y}^*(s^t), \\ \hat{y}(s^t) &= \hat{k}(s^t)^\alpha (\hat{A}(s^t) \hat{l}(s^t))^{1-\alpha}, \\ \hat{y}^*(s^t) &= \hat{k}^*(s^t)^\alpha (\hat{A}^*(s^t) \hat{l}^*(s^t))^{1-\alpha}, \\ \gamma \zeta \hat{k}(s^{t+1}) &= (1 - \delta) \hat{k}(s^t) + \hat{i}(s^t), \\ \gamma \zeta \hat{k}^*(s^{t+1}) &= (1 - \delta) \hat{k}^*(s^t) + \hat{i}^*(s^t), \text{ all } s^t \in S^t,\end{aligned}$$

where the process for \hat{A} and \hat{A}^* is given by (2.46).

In what follows, we substitute the value for adjusted discount $(\beta \gamma^{\eta(1-\theta)})^t$ and instead simply write β . If $\theta = 2$, such adjusted value is given by $\beta \equiv (\beta \gamma^{\eta(1-\theta)})^t = .983$. We also drop the notation with ‘hats’, and write g instead of $\gamma \zeta$. After these simplifications, the resulting stationary planning problem is:

$$\max \left[\sum_t^\infty \beta^t \left[\sum_{s^t \in S^t} \pi(s^t) u(c(s^t)) + \sum_{s^t \in S^t} \pi(s^t) u(c^*(s^t)) \right] \right]$$

subject to

$$\begin{aligned}c(s^t) + i(s^t) &= G(d(s^t), f(s^t)), \\ c^*(s^t) + i^*(s^t) &= G(f^*(s^t), d^*(s^t))\end{aligned}$$

and

$$\begin{aligned}
d(s^t) + d^*(s^t) &= y(s^t), \\
f(s^t) + f^*(s^t) &= y^*(s^t), \\
y(s^t) &= k(s^t)^\alpha (A(s^t)l(s^t))^{1-\alpha}, \\
y^*(s^t) &= k^*(s^t)^\alpha (A^*(s^t)l^*(s^t))^{1-\alpha}, \\
gk(s^{t+1}) &= (1 - \delta)k(s^t) + i(s^t), \\
gk^*(s^{t+1}) &= (1 - \delta)k^*(s^t) + i^*(s^t), \text{ all } s^t \in S^t,
\end{aligned}$$

The competitive equilibrium to this planning solution corresponds exactly to the setup we have described in previous sections (we even have introduced a constant g to take growth into account). So, below, instead of taking the first order conditions to this planning problem, we instead use the equilibrium conditions from decentralized competitive equilibrium that we have stated before. To summarize, these conditions are:

(i) Demand equations (note that this equation embeds numeraire normalization, see derivation of FOC for our prototype model)

$$p_d(s^t) = G_d(d(s^t), f(s^t)), \quad (2.47)$$

$$p_f(s^t) = G_f(d(s^t), f(s^t)), \quad (2.48)$$

$$p_f^*(s^t) = G_f(f(s^t), d(s^t)), \quad (2.49)$$

$$p_d^*(s^t) = G_d(f(s^t), d(s^t)), \quad (2.50)$$

(ii) labor/leisure choice:

$$\frac{u_l(s^t)}{u_c(s^t)} = -w(s^t), \quad (2.51)$$

$$\frac{u_l^*(s^t)}{u_c^*(s^t)} = -w^*(s^t), \quad (2.52)$$

(iii) perfect risk-sharing

$$x(s^t) = \frac{u_c(c^*(s^t))}{u_c(c(s^t))} \quad (2.53)$$

(iv) Euler equations:

$$u_c(s^t) g = \beta E_{s^t}[u_c(s^{t+1}) ((1 - \delta) + r(s^{t+1}))], \quad (2.54)$$

$$u_{c^*}^*(s^t) g = \beta E_{s^t}[u_{c^*}^*(s^{t+1}) ((1 - \delta) + r^*(s^{t+1}))], \quad (2.55)$$

(v) law of one price

$$p_d(s^t) = x(s^t)p_d^*(s^t), \quad (2.56)$$

$$p_f(s^t) = x(s^t)p_f^*(s^t), \quad (2.57)$$

(vi) factor prices:

$$r(s^t) = \alpha p_d(s^t) k(s^t)^{\alpha-1} (A(s^t) l(s^t))^{1-\alpha}, \quad (2.58)$$

$$r^*(s^t) = \alpha p_d^*(s^t) k^*(s^t)^{\alpha-1} (A^*(s^t) l^*(s^t))^{1-\alpha}, \quad (2.59)$$

$$w(s^t) = (1 - \alpha) A(s^t) p_d(s^t) k(s^t)^\alpha (A(s^t) l(s^t))^{-\alpha}, \quad (2.60)$$

$$w^*(s^t) = (1 - \alpha) A^*(s^t) p_d^*(s^t) k^*(s^t)^\alpha (A^*(s^t) l^*(s^t))^{-\alpha}, \quad (2.61)$$

(vii) feasibility and market clearing

$$c(s^t) + i(s^t) = G(d(s^t), f(s^t)), \quad (2.62)$$

$$c^*(s^t) + i^*(s^t) = G(f^*(s^t), d^*(s^t)) \quad (2.63)$$

$$d(s^t) + d^*(s^t) = y(s^t), \quad (2.64)$$

$$f(s^t) + f^*(s^t) = y^*(s^t), \quad (2.65)$$

$$y(s^t) = k(s^t)^\alpha (A(s^t)l(s^t))^{1-\alpha}, \quad (2.66)$$

$$y^*(s^t) = k^*(s^t)^\alpha (A^*(s^t)l^*(s^t))^{1-\alpha}, \quad (2.67)$$

$$gk(s^{t+1}) = (1 - \delta)k(s^t) + i(s^t), \quad (2.68)$$

$$gk^*(s^{t+1}) = (1 - \delta)k^*(s^t) + i^*(s^t), \text{ all } s^t \in S^t. \quad (2.69)$$

(viii) technology shocks

$$\log A(s^{t+1}) = \rho \log A(s^t) + \varepsilon(s^{t+1}), \quad (2.70)$$

$$\log A^*(s^{t+1}) = \rho \log A^*(s^t) + \varepsilon^*(s^{t+1}). \quad (2.71)$$

As we can see, in the system above, we have 12 variables we should count twice, $c, i, y, k, l, A, d, f, r, w, p_d, p_f$, and 1 variable that we should count once, x . Together, it gives us 25 variables. Since we have 25 equations, unless we have mistakenly restated same equilibrium conditions twice (not the case), we can proceed setting up the model on the computer.

To solve the model, we implement the perturbation method using the package Dynare⁴². Dynare will locally approximate the solution around the deterministic steady state, which we next calculate analytically.

⁴²See <http://www.ceprenap.cnrs.fr/dynare/>.

To calculate the steady state, it is convenient to unwind some of the steps used in the calibration. Specifically, we proceed as follows:

Step 1: Under symmetry, $p_f = p_d$, and from demand equations (i), we have

$$f = \left(\frac{\omega}{1-\omega}\right)^{-\sigma} d.$$

From the definition of the import ratio, we obtain

$$is \equiv \frac{f}{d+f} = \frac{\left(\frac{\omega}{1-\omega}\right)^{-\sigma} d}{\left(\frac{\omega}{1-\omega}\right)^{-\sigma} d + d} = \frac{\left(\frac{\omega}{1-\omega}\right)^{-\sigma}}{\left(\frac{\omega}{1-\omega}\right)^{-\sigma} + 1},$$

and calculate

$$\omega = \frac{\left(\frac{1-is}{is}\right)^{\frac{1}{\sigma}}}{\left(\frac{1-is}{is}\right)^{\frac{1}{\sigma}} + 1}. \quad (2.72)$$

Using Euler's law, we next solve for prices $p_d = p_f$ (equal by symmetry). By hod 1 of G , we have

$$p_d d + p_f f = G(d, f),$$

and thus

$$p_d = p_f = \frac{G(d, f)}{d+f} = (\omega \times (1-is)^{\frac{\sigma-1}{\sigma}} + (1-\omega) \times is^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}}, \quad (2.73)$$

where is denotes the import share that we have used in calibration as one of the data targets.

Step 2: Since the share of labor in time endowment of has been calculated to be $l = .329$, knowing that by Euler's equation (iv)

$$r = \frac{g}{\beta} - (1-\delta), \quad (2.74)$$

we can use factor price equation for rental price of capital (ii)

$$r = \alpha p_d \left(\frac{k}{l}\right)^{\alpha-1}$$

to find steady state level of capital⁴³

$$k = l \left(\frac{\alpha p_d}{r}\right)^{\frac{1}{1-\alpha}}. \quad (2.75)$$

Step 3: By definition of is , from feasibility conditions (vii), we have

$$d = (1 - is) \times k^\alpha l^{1-\alpha}, \quad (2.76)$$

$$f = is \times k^\alpha l^{1-\alpha}. \quad (2.77)$$

Step 4: The remaining steady state variables can be calculated from (vi), (vi) and (iii):

$$c = (\omega d^{\frac{\sigma-1}{\sigma}} + (1 - \omega) f^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} - (g - 1 + \delta)k, \quad (2.78)$$

$$w = (1 - \alpha) p_d k^\alpha l^{-\alpha},$$

$$x = 1 \text{ (by symmetry)}. \quad (2.79)$$

Step 5: The implied values of the deep parameter η can be retrieved from the assumed value of l the same way as we have done to calibrate the model. Labor-leisure choice (vi) implies

$$\begin{aligned} w &= -\frac{u_l}{u_c} = \frac{c}{1-l} \frac{1-\eta}{\eta}, \\ \frac{1-\eta}{\eta} &= \frac{w(1-l)}{c} \end{aligned}$$

⁴³We could have alternatively solved for steady state in the closed economy model first, and used the quantities from there to plug in here. This approach would require some adjustment so that $G(d, f) = y$ for steady state values.

and thus

$$\eta = \frac{1}{1 + \frac{w(1-l)}{c}}. \quad (2.80)$$

The numbered equations fully characterize the deterministic steady state we set out to find.

We next proceed with the implementation of the Dynare code⁴⁴ to solve the model using perturbation method. The codes can be downloaded from my website.

Exercise 51 *The example will illustrate (in an abstract way) how to use the perturbation method to solve portfolio choice problems under uncertainty. In the deterministic steady state, any portfolio that gives the same wealth distribution is equivalent. However, this equivalence breaks down when we add uncertainty (shocks), because stochastic payoffs of different portfolios will typically correlate differently with the stochastic consumption. To solve the optimal portfolio problem using perturbation method, the idea is to start with an arbitrary portfolio, solve for the policy for assets with a penalty function imposed (2nd order approximation at least), and use this policy to obtain the next guess. For an implementation of such method, see, for example, Heathcote and Perri (2008): “The International Diversification Puzzle is Not as Bad as You Think”, Minneapolis Fed Staff Report 389. Using an abstract example, we will demonstrate how it works.*

Consider the following problem

$$\max_{x \geq 0} [x^\rho + (1-x)^\rho].$$

Clearly, when $0 < \rho < 1$, there is a unique solution $x = \frac{1}{2}$, and when $\rho = 1$, any solution $x \in [0, 1]$ solves the problem.

Suppose now that we do not know the true solution when $0 < \rho < 1$, but we do know that any value of x satisfies the problem for $\rho = 1$. We will exploit this fact by

⁴⁴http://www.ceprenap.cnrs.fr/juillard/mambo/download/manual/Dynare_UserGuide_WebBeta.pdf

using the perturbation method solve for the case when $\rho < 1$.

a. Take as a starting guess $x_{guess} = 0.25$ and proceed as follows. Impose a convex penalty on the objective function and consider the following artificial problem:

$$\max_{x \geq 0} [x^\rho + (5 - x)^\rho - \phi(x_{guess} - x)^2].$$

Using the idea of perturbation method, implement on Matlab the following scheme. Choose appropriate value of ϕ that is large enough, but not too small, and take 2nd order approximation of the solution to the artificial problem for $\rho = .95$ with $x_{guess} = 0.25$ and perturbation wrt ρ (evaluated at $\rho = 1$). After you solve for a new approximate value, take this value as the next guess x_{guess} , and solve it again. Iterate until convergence.

b. Have you obtained the true solution $x = .5$? Briefly explain what makes this scheme work, and why we needed to impose penalty ϕ (what would go wrong?)?

2.6 Quantitative Comparison of Theory and Data

Our goal here is to obtain the simulated time-series from the models, and after treating this ‘artificial data’ the same way as the actual data, to compare the statistics pertaining to the properties of the business cycle fluctuations. Below, we calculate business cycle statistics implied by 4 different versions of our two-country model (all for $\theta = 2$):

1. Model with capital and labor/leisure choice (model 1) with high elasticity of substitution $\sigma = .73$,
2. Model with capital and labor/leisure choice (model 1) with low elasticity of substitution $\sigma = 8$.

3. Prototype endowment model (model 2) with a high elasticity of substitution
 $\sigma = .73$,
4. Prototype endowment model (model 2) with a low elasticity of substitution
 $\sigma = 8$

In our comparison with data, we will rely on a casual notion of the quality of fit. We should stress that in order to focus on the qualitative economic mechanisms, we have deliberately setup a model that is misspecified in many dimensions. Fitting such model to the data using statistical estimation would likely result in a biased inference about parameter values.⁴⁵ Of course, with more complete models, it is a good idea to try to estimate them too. To learn about the methods of estimating large-scale DSGE models, check out the website of Frank Schorfheide. On the website of Dynare you will also find helpful examples applying Bayesian techniques to estimate a macro model (see the link to examples).

Properties of the Data

Before we proceed any further, we should first characterize how economic activity behaves over the business cycle. Without it, we will not be able to say much about the performance of our model. To characterize properties of economic activity over the business cycle, we will focus attention on two first order aspects of the data: comovement and volatility.

Figure 2.11 lists key summary statistics pertaining to 5 basic measures of aggregate activity over the business cycles for 3 major economic regions of the world: US, Japan and Europe (aggregate of EU15 countries). The 5 measures are: output, consumption expenditures, investment, employment and net exports (borrowing from rest of the world). All measures are real and do not depend on prices. The data is quarterly,

⁴⁵If the omitted variables turn out correlated with the error term, this creates a problem for statistical methods that try to fit the data directly.

| International Comovement | | | | Volatility | | | |
|--------------------------|-------------|------|-------|------------|--------------------|-------|-------|
| Variable | Correlation | | | Variable | Standard deviation | | |
| | US | EU15 | Japan | | US | EU15 | Japan |
| Y | 0.40 | 0.52 | 0.24 | Y | 1.33% | 0.74% | 1.19% |
| A | 0.33 | 0.48 | 0.04 | A | 0.80% | 0.63% | 1.54% |
| C | 0.23 | 0.36 | -0.01 | C | 0.99% | 0.71% | 0.80% |
| I | 0.23 | 0.54 | 0.36 | I | 3.71% | 2.33% | 2.99% |
| L | 0.21 | 0.47 | -0.01 | L | 1.08% | 0.77% | 0.45% |
| G | 0.17 | 0.20 | -0.15 | G | 0.91% | 0.46% | 1.00% |
| C+G | 0.25 | 0.39 | -0.03 | K | 0.35% | 0.22% | 0.33% |

| Domestic Comovement (US data) | | | | | | |
|-------------------------------|-------------|------|------|------|------|-------|
| Variable | Correlation | | | | | |
| | Y | A | C | I | L | NX |
| Y | 1.00 | 0.88 | 0.83 | 0.93 | 0.85 | -0.21 |
| A | | 1.00 | 0.69 | 0.78 | 0.51 | 0.01 |
| C | | | 1.00 | 0.81 | 0.73 | -0.22 |
| I | | | | 1.00 | 0.85 | -0.18 |
| L | | | | | 1.00 | -0.33 |
| NX | | | | | | 1.00 |

Note: Statistics based on quarterly data logged and hp filtered (1600), 1980-2005. Variables: Y - real GDP, A - Solow residual, C - consumption, I - investment, L - employment, G - government consumption. Source: SourceOECD.org.

Figure 2.11: Basic Properties of International Business Cycles: Volatility and Comovement Patterns.

and has been first logged and the HP-filtered (1600). NX has been calculated as the ratio of the difference between nominal exports and nominal imports to nominal GDP (results are the same if CPI is used to deflate NX or trend part of nominal GDP).

As we can see, even though statistics are widely dispersed, several patterns emerge. First, in all three cases the ranking of relative volatilities is similar. Investment is clearly the most volatile, output and Solow residual are second, and consumption is the least volatile. Solow residual is less volatile in US and EU15 than output and more in Japan.⁴⁶

In terms of international comovement, in all three blocks economic activity tends to positively comove with the rest of the world. In Japan, international comovement is the weakest, but Europe positively comoves with the rest of the world very strongly. The US is somewhat inbetween. As far as ranking of comovement across our 4 ag-

⁴⁶At this point we should mention that it is rather standard in the literature to call $A^{1-\alpha}$ a Solow residual. Residual defined in such a way would be less volatile than output in all three economic blocks (not reported here).

gregate measures, the data shows the following pattern. Output, investment, labor and Solow residuals all positively comove, and in particular, they comove more than consumption does. Also, output is most correlated internationally, in particular, more so than the Solow residuals. The low positive comovement of consumption relative to output is somewhat puzzling, because one should expect that consumption risk sharing should make consumption to be most strongly correlated internationally. It actually turns out a wrong intuition when the elasticity of substitution between the domestic and the foreign goods is very low, and there is home-bias (to understand why think about the Leontief case).

Another clear pattern is the countercyclicalities of net exports (NX). It turns out that a country that has a boom tends to import more and borrow from the rest of the world—again the opposite to what simple logic about risk sharing would suggest. Interestingly enough, there is little evidence that Solow residuals are negatively correlated with NX. It is capital and labor that seem to be behind this negative correlation between output and NX.

All these properties, perhaps except the last one, are robust across countries, and have been widely documented in the literature. Finally, we should also mention that all variables are highly persistent, which we do not report here.

Having characterized how economic activity behaves over the business cycle, we next study the predictions of the model. The goal is intuitively understand the economic forces behind the implications of the theory, and organize the findings in a set of discrepancies between the theory and the data. Since our theory is the most basic frictionless environment one could think of, we will refer to these discrepancies as *puzzles wrt to standard theory*, which helps us organize further work to improve upon this theory.

Predictions of the Models

To map the model onto the data, we first have to define consistent with the data system of measurement. In this respect, we identify the following objects in the model with their corresponding counterparts in the data:

- Real GDP = $p_d^{ss}d + p_f^{ss}d + x^{ss}p_d^{ss}d^* - p_f^{ss}f$,
- GDP in current prices = $p_d d + p_f d + x p_d d^* - p_f f$,
- Consumption = $\frac{c}{G(d,f)}(p_d^{ss}d + p_f^{ss})$,
- Investment = $\frac{i}{G(d,f)}(p_d^{ss}d + p_f^{ss})$,
- Net exports (in current prices) = $(x p_d d^* - p_f f)/GDP$,
- Real net exports = $(x p_d^{ss}d^* - p_f^{ss}f)/Real\ GDP$
- Real export price = $x p_d^*$,
- Real import price = p_f ,
- Terms of trade = p ,
- Real exchange rate = x ,

where \cdot^{ss} denotes steady state prices. (To be fully consistent with the data, we should actually measure prices like real exchange rate using fixed weights CPI rather than the ideal one—just like in the data. This would not matter in this model, and so we omit this distinction.)

Table 2.6 illustrates the results implied by the models. As we can see, in terms of prices, all 4 models exhibit the patterns we have discussed. This should not surprise, as the supply-side extensions that we have considered are irrelevant for these facts. Specifically, real exchange rate is about 4 times less volatile than in the data, and

it barely moves when elasticity is high. This is to be expected, since real exchange rate movements come from relative price movements (terms of trade). When goods are closely substitutable, this relative price does not move much. Export and import prices are negative correlated, and terms of trade is more volatile than the real exchange rate. Pretty much everything is the opposite of what it should be.

In terms of quantities, the models fare much better, but only with low elasticity of substitution between goods. The high elasticity case is a disaster. In the case of low elasticity, statistics are in the neighborhood of what they should be. The models predict positive international comovement of economic activity. Of course, statistics do not match up exactly, but we know that there are additional tweaks on the model can further improve the fit (e.g. home production, convex adjustment cost on capital, non-tradable sector, see exercise below).

In all models, there is clearly too little propagation—absolute volatility of GDP is too low. Also, consumption comoves internationally too much, in particular, it is more correlated internationally than output. The volatility of GDP and consumption falls short in terms of the data. Investment, on the other hand, is way too volatile, but just like in the data, it is the most volatile time series and highly procyclical.

As already mentioned, the problems on the quantity side can be fixed by incorporating additional features, especially convex adjustment cost on consumption and home production. With these two features the model can replicate quantities in most dimensions, except for excess international comovement of consumption. However, we should stress that consumption in the data measures consumption expenditures, and does not take into account that durable consumption can be way more volatile. One should thus be careful with the interpretation of statistics pertaining to consumption data.⁴⁷

⁴⁷See, for example, the paper by Charles Engel & Jian Wang, 2008. “International Trade in Durable Goods: Understanding Volatility, Cyclicalities, and Elasticities,” NBER Working Papers 13814, National Bureau of Economic Research, Inc.

Lastly, note that the model with capital quite successfully predicts countercyclicality of NX. In fact, our model goes against simple intuition that during booms, due to risk sharing, country should export more than import and lend to the rest of the world, which requires some explanation. As we can see, two factors are important: low elasticity of substitution (even when there is no capital), and capital.

The reason why capital (and labor leisure choice) plays a role is the production efficiency motive. This motive dictates that economic activity should move to the country with highest productivity. In the words of Backus et al., in these models “one wants to grow *hey where the sun shines*”. As a result, during booms, even though there is a risk sharing motive to ship goods abroad, there is also an offsetting motive to invest in capital at home to take advantage of higher productivity. This results in NX in ‘consumption goods’ going into surplus, but NX in ‘investment goods’ going into deficit. If the second effect dominates, the NX becomes countercyclical. To formalize this idea, recall the setup from exercise (46). In this setup, we can directly decompose net export into ‘consumption component of NX’ and ‘investment component of NX’

$$NX = \underbrace{(p_d D^* - p_f F)}_{NX_c} + \underbrace{(I_d^* - I_f)}_{NX_I}.$$

By the equivalence result you proved in this exercise, such decomposition also applies to our benchmark economy—it is just less transparent in such case. Through the lens of this decomposition, consumption risk sharing motive makes NX_c typically go up, but production efficiency motives offsets these movements through NX_I .

To understand why elasticity is so important, we should look at our endowment economy. In this economy, we can analytically show that when elasticity is 1 NX is actually zero—which also goes against the common wisdom of risk sharing. Why is that? The following calculation, works out this case, which gives a clear intuition:

$$\begin{aligned}
NX &= \text{exports} - \text{imports} = \\
&= xp_d^*(y - \omega y) - p_f(1 - \omega)y^* = \\
&= xp_d^*[(y - \omega y) - \frac{p_f}{xp_d^*}(1 - \omega)y^*],
\end{aligned}$$

where

$$\begin{aligned}
\frac{p_f}{xp_d^*} &= \frac{1 - \omega}{\omega} \frac{d^\omega f^{-\omega}}{d^{\omega-1} f^{1-\omega}} = \frac{1 - \omega}{\omega} \frac{f}{d} = \\
&= \frac{1 - \omega}{\omega} \frac{\omega y}{(1 - \omega)y^*} = \frac{y}{y^*},
\end{aligned}$$

and thus

$$\begin{aligned}
NX &= xp_d^*[y - \omega y - \frac{y}{y^*}(1 - \omega)y^*] = \\
&= xp_d^*[y - \omega y - (1 - \omega)y] = 0.
\end{aligned}$$

Namely, these are not quantities that are countercyclical. During booms, the country ships more goods abroad in terms of physical units (because $d^* = (1 - \omega)y$ and $f = (1 - \omega)y^*$), it is the value of what is exported in terms of what is important that offsets these movements (terms of trade movements).⁴⁸ When the elasticity is low, these movements cause wealth effects that level off NX, and for σ below unity, actually turn it negative.

[ADD IMPULSE RESPONSE FUNCTIONS HERE]

⁴⁸It is of interest to look at constant price NX in the data. It has been done, and the model performs much worse in such case. Still, by including physical capital, the model can account for the data.

Exercise 52 *Implement in Dynare a simple closed economy model with analogous parameter setting. Compare the statistics with the corresponding open economy model.*

Exercise 53 *Download the codes from class website. Solve all 4 models under the assumption of financial autarky (i.e. add equation $NX=0$ instead of perfect risk sharing equation). Compare results to the complete market economy, and discuss them.*

Exercise 54 *(Optional) Implement in Dynare an extended model with a convex adjustment cost on capital. Document the effect of the convex adjustment cost on the statistics discussed above.*

2.7 Drozd and Nosal (2008)

Drozd and Nosal (2008) propose a simple way to reconcile the quantity side of dynamic IRBC theory with a high long-run elasticity, and show that such frictions can also successfully account for the correlations of international prices. The basic idea is that firms, in order to sell their output, need to build the demand and marketing infrastructure. The modeling tool is search theory.

Drozd and Nosal adopt the baseline BKK model and embed their model of marketing into this basic structure. Their model generates a low measured volatility ratio that is consistent with high assumed long-run elasticity σ , and high measured long-run elasticity. At the same time, quantities behave exactly as in the standard model with low elasticity of substitution.

[to be completed]

2.8 Other Extensions

Exotic Elasticities

[to be completed]

Non-tradable Goods

[to be completed]

Home Production

[to be completed]

Durable Consumption

[to be completed]

Habit formation

[to be completed]

Table 2.6: Comparison of Models with Data.

| | | Models | | | |
|--------------------------|-------|-----------------------------------|------------------------------------|-----------------------------------|------------------------------------|
| Statistic | Data | Model 1 with Low Elasticity | Model 1 with High Elasticity | Model 2 with Low Elasticity | Model 2 with High Elasticity |
| International Prices | | | | | |
| A. Correlations | | | | | |
| p_x, p_m | 0.75 | -1.00 | -1.00 | -1.00 | -1.00 |
| p_x, x | 0.46 | -1.00 | -1.00 | -1.00 | -1.00 |
| p_m, x | 0.69 | 1.00 | 1.00 | 1.00 | 1.00 |
| p, x | 0.61 | 1.00 | 1.00 | 1.00 | 1.00 |
| B. Standard deviation | | | | | |
| x | 3.60 | 0.52 | 0.15 | 1.12 | 0.18 |
| - Relative to x | | | | | |
| p_x | 0.37 | 0.17 | 0.17 | 0.17 | 0.17 |
| p_m | 0.61 | 1.17 | 1.17 | 1.17 | 1.17 |
| p | 0.27 | 1.33 | 1.33 | 1.33 | 1.33 |
| Quantities | | | | | |
| A. Correlations | | | | | |
| - Domestic with foreign | | | | | |
| Solow Res. | 0.30 | 0.26 | 0.26 | n.a. | n.a. |
| GDP | 0.40 | 0.34 | -0.02 | 0.36 | 0.36 |
| Consumption | 0.25 | 0.37 | 0.74 | 0.69 | 0.99 |
| Employment | 0.21 | 0.55 | -0.34 | n.a | n.a |
| Investment | 0.23 | 0.24 | -0.46 | n.a. | n.a. |
| - GDP with | | | | | |
| Consumption | 0.83 | 0.95 | 0.86 | 0.98 | 0.86 |
| Employment | 0.85 | 0.94 | 0.97 | n.a. | n.a. |
| Investment | 0.93 | 0.64 | 0.47 | n.a. | n.a. |
| Net exports | -0.49 | -0.57 | -0.08 | -0.57 | 0.57 |
| - Terms of trade with | | | | | |
| Net exports | -0.17 | -0.83 | 0.85 | -1.00 | 1.00 |
| B. Standard deviations | | | | | |
| GDP | 1.33 | 0.87 | 1.01 | 0.80 | 0.80 |
| - Relative to GDP^{**} | | | | | |
| Consumption | 0.74 | 0.45 | 0.34 | 0.89 | 0.81 |
| Investment | 2.79 | 3.03 | 3.97 | n.a. | n.a. |
| Employment | 0.81 | 0.37 | 0.48 | n.a. | n.a. |
| Net exports | 0.29 | 0.13 | 0.40 | 0.02 | 0.47 |

Statistics based on logged and Hodrick-Prescott filtered time series (with $\lambda = 1600$). Data column refers to US data for the time period 1980:1-2004:1.

*Ratio of corresponding standard deviation to the standard deviation of x .

**Ratio of corresponding standard deviation to the standard deviation of GDP .

Chapter 3

Papers for Student Presentations

Suggested papers:

- Atkeson, Andrew & Patrick Kehoe & Fernando Alvarez (2008): “Time-Varying Risk, Interest Rates, and Exchange Rates in General Equilibrium”, Minneapolis FED Staff Report 371, September
- Perri, Fabrizio & Jonathan Heathcote (2007): “The International Diversification Puzzle Is Not as Bad as You Think”, Staff Report 398, October
- Michael Waugh (2007): “International Trade and Income Differences”, University of Iowa, unpublished manuscript
- Arellano, Cristina (2007): “Default Risk and Income Fluctuations in Emerging Economies”, American Economic
- Klette, Tor Jakob & Samuel Kortum (2004): “Innovating Firms and Aggregate Innovation,” Journal of Political Economy, University of Chicago Press, vol. 112(5), pages 986-1018, October
- Mark Aguiar & Manuel Amador & Gita Gopinath, 2005. “Efficient Fiscal Policy and Amplification,” NBER Working Papers 11490, National Bureau of Economic Research, Inc.

- Mark Aguiar & Gita Gopinath, 2007. “Emerging Market Business Cycles: The Cycle Is the Trend,” *Journal of Political Economy*, University of Chicago Press, vol. 115, pages 69-102.