HOMEWORK 1: Solutions

Exercise 1 *Show that in equilibrium the following version of the law of one price must hold:*

$$p_{ni} = d_{ni}p_{ii}, \text{ all } n, i = 1, ..., N.$$

Before we proceed, we prove a helpful lemma. The proof of the lemma is trivial and is omitted.

Solution 2 *Firms sell in equilibrium to all markets. In equilibrium, must satisfy the following profit maximization problem:*

$$\Pi_i = \max \sum_{i=1..N} p_{ni}y_{ni} - w_i l_i. \tag{1}$$

subject to

$$\sum_{i=1..N} d_{ni}y_{ni} \leq l_i. \tag{2}$$

Since the problem is linear, we may have a corner solution (0, or max does not exists). However, neither corner solution is consistent with the definition of equilibrium. So, equilibrium allocation must be an interior solution to the above problem. Using FOC (neccessary condition for interior solution), we obtain the answer.

Lemma 3 *Equilibrium allocation in this economy is homogenous of degree 1 wrt the endowment vector* $(L_n)_n$*, and equilibrium prices are homogenous of degree 0 wrt the endowment vector* $(L_n)_n$.

Exercise 4 *Prove the above lemma.*

Solution 5 *Need to show that equilibrium allocation*

$$\{c_{ni}((L_n)_n), y_{ni}((L_n)_n), l_n((L_n)_n)\}$$

has the following property

$$\{c_{ni}(\lambda(L_n)_n), y_{ni}(\lambda(L_n)_n), l_n(\lambda(L_n)_n)\} = \{\lambda c_{ni}((L_n)_n), \lambda y_{ni}((L_n)_n), \lambda l_n((L_n)_n)\},$$

for any $\lambda > 0$. *So, suppose* $\{c_{ni}((L_n)_n), y_{ni}((L_n)_n), l_n((L_n)_n)\}$ *is an equilibrium allocation in an economy with endowment vector* $(L_n)_n$. *Need to show that* $\lambda c_{ni}((L_n)_n), \lambda y_{ni}((L_n)_n), \lambda l_n((L_n)_n)$ *is an equilibrium allocation in an economy with endowment vector* $(\lambda L_n)_n$. *Equilibrium allocation satisfies the following planning problem for any* $\lambda > 0$:

$$U(\lambda) = \max_{c_{ni}} \left(\sum_{i=1..N} \alpha_i^{\frac{1}{\sigma}} c_{ni}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}},$$

subject to

$$\sum_{i=1..N} d_{ni} c_{ni} = \lambda L_i, \text{ all } n, i.$$

The solution exists and is unique. Furthermore, $U(\lambda)$ *is hod 1. We can show the latter claim as follows. Divide the constraint by* λ *to obtain*

$$U(\lambda) = \max_{c_{ni}} (\sum_{i=1..N} \alpha_i^{\frac{1}{\sigma}} c_{ni}^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}},$$

subject to

$$\sum_{i=1..N} d_{ni} (\frac{c_{ni}}{\lambda}) = L_i, \text{ all } n, i.$$

Define $\hat{c}_{ni} \equiv \frac{c_{ni}}{\lambda}$, *and plug in for* $c_{ni} = \lambda \hat{c}_{ni}$ *to the objective function. Nnote that you can write:*

$$U(\lambda) = \lambda \max_{c_{ni}} (\sum_{i=1..N} \alpha_i^{\frac{1}{\sigma}} (\hat{c}_{ni})^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}},$$

subject to

$$\sum_{i=1..N} d_{ni} \hat{c}_{ni} = L_i, \text{ all } n, i,$$

Thus, by definition of U(1), we obtain

$$U(\lambda) = \lambda U(1).$$

Let $c^*_{ni}(\lambda)$ *denote the maximizer of the above problem, which we know must be unique given* λ *(*$c^*_{ni}(\lambda)$ *is a function). Now, we must show that* $c^*_{ni}(\lambda) = \lambda c^*_{ni}(1)$. *By contradiction, suppose that* $c^*_{ni}(\lambda) \neq \lambda c^*_{ni}(1)$. *The, we have*

$$\begin{aligned} U(\lambda) &> \max_{c_{ni}} (\sum_{i=1..N} \alpha_i^{\frac{1}{\sigma}} (\lambda c^*_{ni}(1))^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} = \lambda \max_{c_{ni}} (\sum_{i=1..N} \alpha_i^{\frac{1}{\sigma}} (c^*_{ni}(1))^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} \\ &= \lambda U(1). \end{aligned}$$

This is a contradiction, because $U(\lambda)$ *is hod 1.*

Exercise 6 *Consider the following expenditure minimization problem:*

$$E(U) = \min_{(c_i)_{1..N} \geq 0} \sum_{i=1}^{N} p_i c_i$$

subject to

$$\begin{aligned} \sum_{i=1}^{N} (\alpha_i^{\frac{1}{\sigma}} c_i^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} &= U, \\ c_i &\geq 0, \text{ all } i = 1..N, \end{aligned}$$

where $p_i's$ denote prices, $E(U)$ are total expenditures (given U), $\alpha_i's$ are the preference weights, σ is the elasticity of substitution, and U is the 'composite good' consumption level (or simply utility). Assume that $p_i's, \alpha_i's, \sigma$ and U are all strictly positive.
a. Show that $E(U)$ is homogenous of degree 1 ($E(\mu U) = \mu E(U)$, all $\mu > 0$), and thus takes the form $P \times U$ where $P = E(1)$.
b. Prove the Envelope Theorem in the context of the problem stated in (a), i.e. show that $E'(U) = \lambda$, where λ is the Lagrange multiplier on the constraint in (a). Then, use the conclusion from point (a) to say $E'(U) = P$, and thus by Envelope Theorem to say $\lambda = P$. Using it, solve for $E(1)$, which together with (a) shows

$$E(U) = (\sum_{i=1}^{N} \alpha_i p_i^{1-\sigma})^{\frac{1}{1-\sigma}} U.$$

(This is an alternative way of deriving the price index to the one we did in the proof of the proposition above.)
c. Show that the expenditures minimization problem with $E(U)$ =Income, is equivalent the underlying utility maximization problem given by:

$$U = \max_{c_i \geq 0} (\sum_{i=1..N} \alpha_i^{\frac{1}{\sigma}} c_i^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}},$$

subject to

$$\sum_{i=1...N} p_i c_i = \textit{Income}.$$

***Solution 7** a. In the solution to the previous exercise we have demonstrated that. Same reasoning applies.*

b. Note that we can drop non-negativity constraint and consider the following lagrangian:

$$L = \sum_{i=1}^{N} p_i c_i - \lambda(\sum_{i=1}^{N} (\alpha_i^{\frac{1}{\sigma}} c_i^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} - U),$$

FOC is a neccesary condition for solution.

Now, consider the following simplified problem

$$L(c, U) = F(c) - \lambda(G(c) - U),$$

Let $c^*(b)$ be the solution to this problem. Then, it must be true that for all U

$$\begin{aligned} \frac{\partial F(c^*(U))}{\partial c_i} &= \lambda \frac{\partial G(c^*(U))}{\partial c_i}, \text{ all } i \\ G(c^*(U)) &= U. \end{aligned}$$

Now, evaluate the following total derivative

$$\frac{dL(c^*(U); U)}{dU} = \sum_i \frac{\partial F(c^*(U))}{\partial c_i} \frac{dc_i^*(U)}{dU} - \lambda \sum_i \frac{\partial G(c^*(U))}{\partial c_i} \frac{dc_i(U)}{dU} + \lambda.$$

By the first order condition,

$$\sum_i \frac{\partial F(c^*(U))}{\partial c_i}\frac{dc_i^*(U)}{dU} = \lambda \sum_i \frac{\partial G(c^*(U))}{\partial c_i}\frac{dc_i(U)}{dU},$$

and thus

$$\frac{dL(c^*(U);U)}{dU} = \lambda,$$

and since also by first order condition

$$L(c^*(U);U) = F(c^*(U)), \ \textit{all } U,$$

it must be that

$$\frac{dL(c^*(U);U)}{dU} = \frac{dF(c^*(U);U)}{dU},$$

and thus

$$\frac{dF(c^*(U);U)}{dU} = \lambda.$$

Finally, note that by hod 1 $E(U) = UE(1)$, *and so* $E'(U) = E(1)$. *But, since by Envelope Thm we have* $E'(U) = \lambda$, *where* λ *is the lagrange multiplier on the constraint set, we also have* $E(1) = \lambda$. *Using this fact, it is easy to solve for E(1). Note that E(1) is given by*

$$E(1) = \sum_{i=1}^{N} p_i c_i^*$$

where c_i^* *can be solved for from the following Lagrangian*

$$L = \sum_{i=1}^{N} p_i c_i - E(1)(\sum_{i=1}^{N}(\alpha_i^{\frac{1}{\sigma}} c_i^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} - 1).$$

It is now straightforward to obtain

$$E(1) = (\sum_{i=1}^{N} \alpha_i p_i^{1-\alpha})^{\frac{1}{1-\alpha}}.$$

(This is a bit long way to solve for the price index, but it nicely illustrates some key dual properties of maximization problems with function hod 1).

Exercise 8 *Suppose that the preferences of the household are instead described by:*

$$U_n = (C^{NT})^{\gamma}(\sum_{i=1..N} \alpha_i^{\frac{1}{\sigma}} c_{ni}^{\frac{\sigma-1}{\sigma}})^{\frac{(1-\gamma)\sigma}{\sigma-1}},$$

where C^{NT} *is the consumption of the local non-tradable good (services). Assume that production technology of the non-tradable good is linear and assume that labor is perfectly mobile across the two sectors. In the extended model, derive the gravity equation by modifying each step in the above proof accordingly.*

Solution 9 *Here, it suffices to notice that household will spend γ fraction of their income, $w_i L_i$, on the non-tradable good, and 1-γ on tradable good. Given spending on the tradable goods, the solution to the household's problem must satisfy:*

$$U_n = \max(\sum_{i=1..N} \alpha_i^{\frac{1}{\sigma}} c_{ni}^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}},$$

subject to

$$\sum_{i=1...N} p_{ni} c_{ni} = (1-\gamma) w_i L_i.$$

Otherwise, it would not solve the original problem. Since , in the derivation of gravity equation we have used definition of expenditures and FOC to household problem, it suffices to to replace X_n with $\hat{X}_n \equiv (1-\gamma)X_n$, and everything will go through.

Exercise 10 *(Numerical experiment) Consider the Armington model with the following settings: $N = 100$, $\alpha = 1/N$, $\sigma = 11$, $L_i = L_j = 1$, all $i, j = 1..N$, where N denotes number of regions in the world. Assume these are regions in two countries: US (large country) and Canada (small country), where 10 regions are in Canada 90 are in the US (roughly corresponds to the ratio of Canadian to US GDP). Furthermore, assume that the transportation cost between the regions within the same country is zero, i.e. $d_{ni} = 1$ whenever $i, n \in US$, or $i, n \in CA$, and assume that the iceberg transportation cost between the regions within two different countries is 20%, i.e. $d_{ni} = 1.2$ whenever $i \in US$, $n \in CA$ or $i \in CA$, $n \in US$.*
a. Use the following equation

$$p_{ii} L_i = \sum_n X_{ni} = \sum_n \alpha_i (\frac{X_{ni}}{X_n}) X_n = \sum_n \alpha_i (\frac{d_{ni} p_{ii}}{P_n})^{1-\sigma} p_{nn} L_n$$

to construct an iterative algorithm that solves the model in MATLAB.[1] *Using this algorithm, compute the overall price level of a representative US region and Canadian region, and prices of the corresponding goods. (HINT: The algorithm may be unstable unless you slow down the updates a bit. To be on the safe side, I suggest to divide both sides by $L_i p_{ii}^{1-\sigma}$, compute p_{ii}, and use the updating rule that puts .5 weight on the old value and only .5 weight on the newly solved value:*[2] *$p_{i+1} = .1p' + .9p_i$, where i is the iteration number, and p_i is used to solve for the vector p' in iteration i. Don't forget to evaluate the convergence, and the residuals of equilibrium conditions at the end. Remember that p_{NN} is the numeraire. (Print out the code and hand in with the HW.)*
b. What is the home-bias from the US side (report $X_{US,US}/X_{US,CAN}$ and interpret) and from the Canadian side (report $X_{CA,CA}/X_{CA,US}$ and interpret).

[1] If it is a contraction, then it will converge to the fixed point.
[2] This way you enlarge the domain on which out mapping is contraction. You then do not need a very precise guess for convergence to the fixed point to occur.

c. Using data generated by the model, suppose you run the following regression of trade flows on the border dummy from the Canadian side only (the way McCallum did in his 1995 paper; includes CA-US, CA-CA observations and excludes US-CA observations):

$$\ln \frac{X_{ni}}{X_i X_n} = \kappa + A \times border\_dummy + \varepsilon,$$

where $n \in CA$, $i \in CA$ or US. What is the value of the regression coefficient on the border dummy?
d. Suppose you run the same regression as in point c but from the US side. What is the value of the regression coefficient on the border dummy?
e. How do your answers to c and d compare to the coefficients that Anderson and Wincoop found in the data by running McCallum's regression separately from the US side and the Canadian side? Explain briefly the implications of your findings.
f. Redo points c and d with $\sigma = 8$.
g. Comparing the answers in e and f, what fraction of the border effect is accounted for by the endogenous multilateral resistance term?
h. What is the average share of trade with the US for a representative Canadian province in the model (measure it by $(90X_{CA,US})/(10X_{CA,CA} + 90X_{CA,US})$)? Consider two levels of trade cost: $d_{US,CA} = 1.2$ (same as before), and $d_{US,CA} = 1.175$. Given that the median and average value of this object in the data is about[3] .45, which level of the border cost accounts better for this number?
i. Would the answers to b-g change if instead you had 100 Canadian regions and 900 US regions? Explain your answer analytically.

Solution 11 *Here is the code that answers all the questions:*

```
%Econ 871 (Numerical experiment with Armington model)
close all
clear
clc
%PARAMETER VALUES:
sig=11; %elasticity of substitution between domestic and foreign goods
N=100; %total number of regions
M=10; %number of regions that are in the small country
d=1.2; %iceberg transportation cost of crossing national border (there is no cost between regions)
%L normalized to 1
alfa=1/N; %weight of each good
%Define matrices
Xni=[]; Xn=[]; cni=[];
%NOTATION
%1 denotes a representative region in small country
```

[3] Pulled out form the data available from Anderson's website.

```
%2 denotes a representative region in large country
%INITIAL GUESS
CPI(1:2)=1.0; p=1.0; p_new=1.0;
%MAIN LOOP THAT SOLVES FOR PRICES USING THE EQUATION STATED IN POINT a
update=1.0;
while (update>10^(-15)) %Main loop that solves the model
 CPI=[(M*alfa*p^(1-sig)+(N-M)*alfa*d^(1-sig) )^(1/(1-sig));
 (M*alfa*(d*p)^(1-sig)+(N-M)*alfa )^(1/(1-sig))]; %from equation for aggregate price index
 p_new = ( M*alfa*(1/CPI(1))^(1-sig)*p+(N-M)*alfa*(d/CPI(2))^(1-sig) )^(1/sig); %from equation for fixed point iterations, notation: p11=p, p22=1.0 (numeraire)
 p=0.8*p+0.2*p_new;
 update=(p-p_new)^2;
end
Xni=[]; Xn=[]; cni=[];
%SOLVE FOR OTHER OBJECTS
%Solve for expenditures
Xni(2,1)=alfa*(d*p/CPI(2))^(1-sig)*1.0;
Xni(1,1)=alfa*(p/CPI(1))^(1-sig)*p;
Xni(1,2)=alfa*(d/CPI(1))^(1-sig)*p;
Xni(2,2)=alfa*(1.0/CPI(2))^(1-sig)*1.0;
%Solve for quantities
cni(1,1)=Xni(1,1)/p;
cni(1,2)=Xni(1,2)/p;
cni(2,1)=Xni(2,1);
cni(2,2)=Xni(2,2)/1.0;
Xn(1)=p;
Xn(2)=1.0;
%PRINT THE RESULTS
disp(sprintf('Convergence achieve at precision %g', update));
disp(sprintf('AGGREGATE PRICE LEVEL'));
disp(sprintf('Price level in the small country = %g', CPI(1)));
disp(sprintf('Price level in the large country = %g', CPI(2)));
disp(sprintf('PRICES OF GOODS'));
disp(sprintf('Price of small country good presented to small country = %g', p));
disp(sprintf('Price of small country good presented to large country = %g', p*d));
disp(sprintf('Price of large country good presented to small country = %g', d));
disp(sprintf('CONSUMPTION OF GOODS'));
disp(sprintf('Quantity of each large country good consumed by a region in small country = %g', cni(1,2)));
disp(sprintf('Quantity of each small country good consumed by a region in large country = %g', cni(2,1)));
disp(sprintf('Quantity of each small country good consumed by a region in small country = %g', cni(1,1)));
```

```
disp(sprintf('Quantity of each large country good consumed by a region in large country = %g', cni(2,2)));
disp(sprintf('HOME-BIAS'));
disp(sprintf('Xni(CA,CA)/Xn(CA)*100 = %g', Xni(1,1)/Xn(1)));
disp(sprintf('Xni(US,US)/Xn(US)*100 = %g', Xni(2,2)/Xn(2)));
disp(sprintf('X(CA,US)/X(CA,CA)*100 = %g', Xni(1,2)/Xni(1,1)));
disp(sprintf('X(US,CA)/X(US,US)*100 = %g', Xni(2,1)/Xni(2,2)));
disp(sprintf('X(CA)/X(US)*100 = %g', Xn(1)/Xn(2)));
disp(sprintf('REGRESSIONS'));
disp(sprintf('CA province trades %g time more with another province than with any US state', exp(log(( Xni(2,2)/(Xn(2)*Xn(2))))-log( ( Xni(2,1)/(Xn(2)*Xn(1)))))));
disp(sprintf('US state trades %g time more with another US state than with any CA province', exp(log(( Xni(1,1)/(Xn(1)*Xn(1))))-log( ( Xni(1,2)/(Xn(2)*Xn(1)))))));
```

Output from the program for $\sigma = 11$:

Convergence achieve at precision 7.81634e-016

AGGREGATE PRICE LEVEL

Price level in the small country = 1.103

Price level in the large country = 1.00652

PRICES OF GOODS

Price of small country good presented to small country = 0.920151

Price of small country good presented to large country = 1.10418

Price of large country good presented to small country = 1.2

CONSUMPTION OF GOODS

Quantity of each large country good consumed by a region in small country = 0.00430469

Quantity of each small country good consumed by a region in large country = 0.00396096

Quantity of each small country good consumed by a region in small country = 0.0612578

Quantity of each large country good consumed by a region in large country = 0.010671

HOME-BIAS

Xni(CA,CA)/Xn(CA)*100 = 0.0612578

Xni(US,US)/Xn(US)*100 = 0.010671

X(CA,US)/X(CA,CA)*100 = 0.0702716

X(US,CA)/X(US,US)*100 = 0.371189

X(CA)/X(US)*100 = 0.920151

REGRESSION

CA province trades 2.47893 time more with another province than with any US state

US state trades 15.4654 time more with another US state than with any CA province

Output from the program for $\sigma = 8$:

Convergence achieve at precision 9.97933e-016

AGGREGATE PRICE LEVEL

Price level in the small country = 1.12233

Price level in the large country = 1.0067

PRICES OF GOODS

Price of small country good presented to small country = 0.909247

Price of small country good presented to large country = 1.0911

Price of large country good presented to small country = 1.2

CONSUMPTION OF GOODS

Quantity of each large country good consumed by a region in small country = 0.0062601

Quantity of each small country good consumed by a region in large country = 0.00569197

Quantity of each small country good consumed by a region in small country = 0.0436591

Quantity of each large country good consumed by a region in large country = 0.0104787

HOME-BIAS

Xni(CA,CA)/Xn(CA)*100 = 0.0436591

Xni(US,US)/Xn(US)*100 = 0.0104787

X(CA,US)/X(CA,CA)*100 = 0.143386

X(US,CA)/X(US,US)*100 = 0.543196

X(CA)/X(US)*100 = 0.909247

REGRESSION

CA province trades 1.67388 time more with another province than with any US state

US state trades 7.67029 time more with another US state than with any CA province

Solution 12 *i. Need to show here that the system you solve on the computer does not depend on N when the ratio of CA to US provinces is kept constant. Namely, need to show that the system is hod 0 wrt N when we keep number of US to CA constant at 10%.*

Exercise 13 *To formalize the argument in text by solving for the demand from the following problem:*

$$\max(q_A^{\frac{\sigma-1}{\sigma}} + Nq_B^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}}$$

subject to

$$p_A q_A + N p_B q_B = I.$$

Specifically, derive the demand for good B, and show that for large N the price index will be affected by the price p_B – implying a lower measured price elasticity of demand. HINT: Derive an equation analogous to (see text). Calculate the price index when N is infinite.

Solution 14 *Price index is*

$$P = (p_A + N p_B^{1-\sigma})^{\frac{1}{1-\sigma}},$$

and demand is

$$q_B = {p_B}^{-\sigma} P^{\sigma-1} I.$$

Definition of elasticity is

$$\varepsilon = -\frac{d\log q_B}{d\log p_B} = -\frac{dq_B}{dp_B}\frac{p_B}{q_B}(= -\frac{\frac{\Delta q}{q}}{\frac{\Delta p}{p}}).$$

Thus,

$$\begin{aligned} \varepsilon(N) &= -\frac{d\log q_B}{d\log p_B} = -\frac{d}{d\log p_B}(I + (\sigma-1)\log P - \sigma \log p_B) = \\ &= (1-\sigma)\frac{d\log P}{d\log p_B} + \sigma \log p_B. \end{aligned}$$

Next, calculate

$$\begin{aligned} \frac{d\log P}{d\log p_B} &= \frac{d}{d\log p_B}[\frac{1}{1-\sigma}\log(p_A^{1-\sigma} + N e^{(1-\sigma)\log p_B})] = \\ &= \frac{N e^{(1-\sigma)\log p_B}}{p_A + N p_B^{1-\sigma}} = \frac{N p_B^{1-\sigma}}{(p_A + N p_B^{1-\sigma})}. \end{aligned}$$

Substituting out, we have

$$\varepsilon(N) = (1-\sigma)\frac{N p_B^{1-\sigma}}{(p_A + N p_B^{1-\sigma})} + \sigma.$$

Taking limit, $N \to \infty$, we obtain

$$\lim_{N\to\infty} \varepsilon(N) = 1.$$

Exercise 15 *(Dornbusch, Fisher and Samuelson (1977)) Consider a world with two* ***symmetric*** *countries and a continuum of goods indexed on a unit interval. Preferences in each country are identical and given by*

$$U_i = (\int_0^1 \ln c_i(\omega) d\omega, \ i = 1, 2,$$

and all markets are perfectly competitive. Assume each country has access to a linear technology to produce each good using labor,

$$y_i(\omega) = z_i(\omega) l_i(\omega),$$

where $z_i(\omega)$ is the efficiency level in producing good ω in country i, and $l_i(\omega)$ is the labor input. Assume that the labor endowment of the stand-in household in each country is one, and the production efficiency schedules are given by the following functions:

$$\begin{aligned} z_1(\omega) &= e^{1-\omega}, \\ z_2(\omega) &= e^{\omega}. \end{aligned} \tag{3}$$

In addition, assume there is a positive tariff rate T between the two countries that amounts to 10% of the value of the transported goods across the border. The revenue from the tariff is lump-sum rebated to the households.
a. Define competitive equilibrium for this economy.
b. ***Refers to point a above.*** *Compute the competitive equilibrium you have defined in a. HINT: Find 2 cutoffs that divide the space of goods into 3 categories: (i) traded and produced in country 1, (ii) traded and produced in country 2, and (iii) not traded (both countries produce them for home market only). Exploit symmetry to say that wages must be 1 in both countries. Use the fact that in the case of log utility the share of expenditures on each good is always a constant fraction of total expenditures on all goods.*
c. Apply NIPA rules to compute the GDP of each economy. What happens to the GDP in equilibrium when the tariffs are increased? HINT: Read handbook of NIPA accounting available from BEA website.[4]
d. How would you have to modify the assumed efficiency schedules stated in (3) to effectively obtain a symmetric two-country Armington model. Based on your answer, what is the key qualitative difference between the Armington model and the DFS model.

Solution 16 *a. Definition of equilibrium is standard: allocation and prices such that everyone maximizes and markets clear.*

b. Note that symmetry implies that we can normalize wages to 1, and thus marginal cost of producing good ω in country 1 is given by $e^{\omega-1}$, and in country 2 by e^{ω}. Now, this allows us to define competitive price $p_{ni}(\omega)$ of good ω from country i presented by country n, which is given by:

$$\begin{aligned} p_{11}(\omega) &= e^{\omega-1}, \\ p_{21}(\omega) &= (1+T)e^{\omega-1}, \\ p_{22}(\omega) &= e^{-\omega}, \\ p_{12}(\omega) &= (1+T)e^{-\omega}. \end{aligned}$$

In this model, the lowest cost country takes over the supply due to perfect competition. We can plot these prices on one diagram, and see that they defines 3 disjoint sets of goods with 2 cutoff values

$$\begin{aligned} \Omega_1 &= \{\omega : p_{21}(\omega) \geq p_{22}(\omega)\}, \\ \Omega_{NT} &= \{\omega : p_{11}(\omega) \leq p_{12}(\omega) \text{ and } p_{11}(\omega) \geq p_{12}(\omega)\}, \\ \Omega_2 &= \{\omega : p_{12}(\omega) \leq p_{11}(\omega)\}, \end{aligned}$$

[4] See http://www.bea.gov/national/pdf/NIPAhandbookch1-4.pdf.

where Ω_i are goods produced only by country i (both for home market and foreign market, and Ω_{NT} are goods that are not trade (produced only for home market) in both countries.

The 1st cutoff value is:

$$(1+T)e^{\omega-1} = e^{-\omega},$$

which after taking logs, gives

$$\begin{aligned} \log(1+T) + \omega - 1 &= -\omega \\ \omega &= \frac{1-\log(1+T)}{2} \simeq \frac{1-T}{2}, \end{aligned}$$

(note that d can not be too high, because there is be no trade). The second cutoff value is:

$$\begin{aligned} (1+T)e^{-\omega} &= e^{\omega-1} \\ \omega &= \frac{\log(1+T)+1}{2} \simeq \frac{1+T}{2}. \end{aligned}$$

Calculating consumption is straintforward. The Lagrangian gives:

$$c_1(\omega) = \frac{1}{\lambda p(\omega)},$$

and budget constraint gives

$$\begin{aligned} \int p(\omega)\frac{1}{\lambda p(\omega)} &= w_1 + T_1, \\ \lambda &= \frac{1}{w_1 + T_1}, \end{aligned}$$

and thus

$$c_1(\omega) = \frac{w_1 + T_1}{p(\omega)},$$

where prices are defined by costs for the respective intervals defining who produces what. Rest of the allocation can be calculated easily

c. GDP is measured in prices that include taxes on production and imports as far as consumption, investment and government expenditures are concerned. However, imports is measured without tariffs, and so tariffs are not subtracted from C+I+G-X-IM in the formula GDP=C+I+G-X-IM. You can see the consequence in GDI tables, there tariffs are actually added to income to be consistent with expenditure side.

d. The efficiency schedules would have to be constant on $\omega = [0, 1/2]$, and 0 on the rest. The other country by symmetry.

Exercise 17 *(Hecksher-Ohlin model) Consider the world with 2 countries and 2 tradable goods. Preferences of the stand-in household in each country are*

$$U_i = \sum_{j=1,2} \log C_i^j, \ i = 1, 2$$

where C_i^j denotes consumption in country i of good j. The stand-in household in country 1 has 2 units of labor (L) and 3 units of capital (K), and the stand-in household in country 2 has 3 units of labor and 2 units of capital. Firms in each country have access to the same CRS technology to produce both goods. The technology to produce good 1 is $Y = K^{\frac{1}{3}} L^{\frac{2}{3}}$ (sector 1), and good 2 is $Y = K^{\frac{2}{3}} L^{\frac{1}{3}}$ (sector 2). For simplicity assume there is no transportation cost.
a. Assume factors are perfectly mobile across countries. Define the competitive equilibrium.
*b. **Refers to equilibrium defined in a.** It can be shown, using the First and the Second Welfare Theorems, that the competitive equilibrium is unique up to the undetermined allocation of capital and labor across countries and sectors, and it solves the following planning problem for $\mu = \frac{1}{2}$:*

$$\max_{(C_i^j, K_i^j, L_i^j)_{i,j=1,2}} \mu \sum_{j=1,2} \log C_1^j + (1-\mu) \sum_{j=1,2} \log C_2^j$$

subject to

$$\begin{aligned} \sum_{i,j=1,2} K_i^j &= 5, \ \sum_{i,j=1,2} L_i^j = 5, \\ \sum_{i=1,2} C_i^1 &= \sum_i \left(K_i^1\right)^{\frac{1}{3}} \left(L_i^1\right)^{\frac{2}{3}}, \\ \sum_{i=1,2} C_i^2 &= \sum_i \left(K_i^2\right)^{\frac{2}{3}} \left(L_i^2\right)^{\frac{1}{3}}. \end{aligned}$$

Compute the competitive equilibrium you defined in a. (HINT: Step 1. Show that $C_i^1 = C_i^2$ all i from the symmetry of the planning problem above. Step 2: Note that the relative price of good 1 versus good 2 is one from MRS. Step 3: The ratio of unit production costs you found in problem 2 is MRT, and since MRS=MRT in equilibrium, find w/r. Step 5: Find $C_i^{j\prime}$s by exploiting symmetry.)
c. Assume factors are immobile across countries. Define the competitive equilibrium.
*d. **Refers to equilibrium defined in c.** It can be shown that there is a unique competitive equilibrium, and it solves the following planning problem for $\mu = \frac{1}{2}$:*

$$\max_{(C_i^j, K_i^j, L_i^j)_{i,j=1,2}} \mu \sum_{j=1,2} \log C_1^j + (1-\mu) \sum_{j=1,2} \log C_2^j$$

subject to

$$
\begin{aligned}
(*)\ \sum_{j=1,2} K_1^j &= 3,\ \sum_{i,j=1,2} L_1^j = 2,\ \sum_{j=1,2} K_2^j = 2,\ \sum_{j=1,2} L_2^j = 3, \\
\sum_{i,j=1,2} K_i^j &= 5,\ \sum_{i,j=1,2} L_i^j = 5, \\
\sum_{i=1,2} C_i^1 &= \sum_i \left(K_i^1\right)^{\frac{1}{3}} \left(L_i^1\right)^{\frac{2}{3}}, \\
\sum_{i=1,2} C_i^2 &= \sum_i \left(K_i^2\right)^{\frac{2}{3}} \left(L_i^2\right)^{\frac{1}{3}}.
\end{aligned}
$$

*Compute the competitive equilibrium you defined in point c. (**HINT**: Guess that the solution to the planning problem above solves a relaxed problem with (*) constraints omitted (like in the planning problem in point b). Verify the guess by showing that factor markets clear – use (*) in combination with the factor demand functions you derived in problem 2 to find market clearing production pattern (write it in matrix form, will be easier...).*
e. What is the pattern of trade in the competitive equilibrium you found in d? More precisely, in which good the labor abundant country is a net exporter?
f. Note that 'the trick' you used in d to solve for the equilibrium would not work in general, i.e. for an arbitrary distribution of factor endowment levels[5]. Show which step of your solution in e would break down if this was not true, and explain why. HINT: Recall that there are non-negativity constraints on all the variables.

Solution 18 *a. Households solve:*

$$
\begin{aligned}
&\max \sum_j \ln C_i^j \\
&\quad s.t. \\
\sum_j p_j C_i^j &= rK_i + wL_i,
\end{aligned}
$$

where $K_1 = 3$, $L_1 = 2$, *and* $K_2 = 2$ *and* $L_2 = 3$.

Firms solve

$$
\max_{k_i^j, l_i^j} \sum_j [p_j Y_i^j - r k_i^j - w l_i^j],
$$

subject to

$$
Y_i^j = k_i^{j\alpha} l_i^{j1-a}.
$$

Markets clear

$$
\sum_j k_i^j = K_i, \sum_j l_i^j = L_i, C_i^j = Y_i^j,
$$

[5] The range of endowment vectors for which 'the trick' works is referred to as the cone of diversification.

all i, j.

b. Symmetry implies $C = C_i^1 = C_i^2$, $w_1 = w_2 = 1$, $r_1 = r_2 = r, w = r$. Due to constant returns to scale, the exact allocation of labor and capital across countries is, however undetermined. It may be that country 1 produces only good 1 and country 2 only good 2, or both goods are produced in each country. Thus, we have to be careful and pin down how factors are allocated between sectors on the aggregate level (world level). So, define $K^1 \equiv K_1^1 + K_2^1$ and $L^1 \equiv L_1^1 + L_2^1$, and note that on the world level Paret optimality requires

$$K^1, L^1 = \arg\max_{K,L}\{K^{\frac{1}{3}}L^{\frac{2}{3}} + p(5-K)^{\frac{2}{3}}(5-L)^{\frac{1}{3}}\},$$

where p is the relative price between commodity 1 and 2, here, by symmetry, p=1. Solving the above problem, we obtain

$$K = \frac{5}{3}, L = \frac{10}{3}.$$

Consumption, can readily be calculated from symmetry and total production. Each country consumes $\frac{5}{32^{\frac{1}{3}}}$ of each good.

c. By analogy to a with restrictions on mobility of factors. We saw in the previous problem that allocation who produces what is undetermined. So, we are free to choose how much of each good is produced in each country. Let Y_1^1 and Y_1^2 be the output in each sector in country 1. Then, we solve for factor demand functions that for a Cobb-Douglas production function are given by

$$\begin{aligned} K(Y; r, w) &= (\frac{\alpha w}{(1-\alpha)r})^{1-\alpha} Y, \\ L(Y; r, w) &= (\frac{1-\alpha r}{\alpha w})^{1-\alpha} Y. \end{aligned}$$

Thus, for the condition (*) to be satisfied, we must have that the implied demand for capital in both sector equals the supply (3 units),

$$(\frac{\alpha_1 w}{(1-\alpha_1)r})^{1-\alpha} Y_1^1 + (\frac{(1-\alpha_2)w}{\alpha_2 r})^{1-\alpha} Y_1^2 = 3,$$

and demand for labor in both sectors equals 2 units,

$$(\frac{(1-\alpha_1)r}{\alpha_1 w})^{1-\alpha} Y_1^1 + (\frac{\alpha_2 r}{(1-\alpha_2)w})^{1-\alpha} Y_1^2 = 2,$$

where $\alpha_1 = \frac{1}{3}, \alpha_2 = \frac{2}{3}$. Plugging in $\frac{w}{r} = 1$, and $\alpha_1 = \frac{1}{3}, \alpha_2 = \frac{2}{3}$, and rewriting in matrix form, we obtain

$$\begin{bmatrix} (\frac{1}{2})^{\frac{2}{3}} & 2^{\frac{1}{3}} \\ 2^{\frac{1}{3}} & (\frac{1}{2})^{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} Y_1^1 \\ Y_1^2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} Y_1^1 \\ Y_1^2 \end{bmatrix} = \begin{bmatrix} (\frac{1}{2})^{\frac{2}{3}} & 2^{\frac{1}{3}} \\ 2^{\frac{1}{3}} & (\frac{1}{2})^{\frac{2}{3}} \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}2^{\frac{2}{3}} \\ \frac{4}{3}2^{\frac{2}{3}} \end{bmatrix}.$$

f. You can clearly see what may go wrong here. Matrix inversion may give negative entries in the inverted matrix, and then the non-negativity condition would be violated. It is just not feasible to produce a negative amount of output.

Intuitively, in this model countries try to achieve optimal allocation you solved in a-b by trading goods and allocating production across the border so that factor demands are consistent with the endowments (to avert lack of factor mobility). But, to support such allocation, sometimes a country would have to sell a negative amount of some good to soak up excess demand for a factor in the other sector. This is, of course, not possible.

Given the above approach to solve the HH's problem that involves and integral, derive the formula for the price index P_n stated above.

Solution 19 *Need to setup the Lagrangian with a variation, and take derivatives. Given FOC, we just proceed as in the Armington model, but with integrals instead of summations. The lagrangian is:*

$$L_\varepsilon = \int_0^\infty p\{c_n(p) + \varepsilon dc_n(p)\} dG_n(p) - \frac{1}{P}\int_0^\infty \{c_n(p) + \varepsilon dc_n(p)\}^{\frac{\sigma-1}{\sigma}} dGn(p) - 1.$$

Taking derivative wrt ε, and evaluating at $\varepsilon = 0$, we obtain

$$p = \frac{1}{P}\frac{\sigma-1}{\sigma} c_n(p)^{-\frac{1}{\sigma}} \ \ a.s.$$

From this equation, we calculate λ analogously to Armington model.

Exercise 20 *Derive the formula for $v_{ni}(z)$ stated above from the underlying unit cost minimization problem, and calculate the value of A so that $c_i = w_i^\beta P_i^{1-\beta}$.*

Solution 21 *Obvious.*

Exercise 22 *Derive the formula for $G_n(p)$.*

Solution 23 *Need to evaluate*

$$G_n(p) = \Pr(p_n \le p) = 1 - \Pr(\mathcal{V}_n \ge p, \ \ all\ n) = 1 - \Pi_n \Pr(\mathcal{V}_n \ge p).$$

$\left[\Gamma(\frac{\theta+1-\sigma}{\theta})\right]^{\frac{1}{1-\sigma}}$. (We need to assume $\theta + 1 - \sigma > 0$; otherwise the above integral is not be well defined.)

Exercise 24 *Show that the mean of the Frechet distribution is $T^{\frac{1}{\theta}}\Gamma(1 - \frac{1}{\theta})$.*

Solution 25 *Need to proceed by analogy to the calculation of price index. Calculate*

$$E(z) = \int_0^\infty F(Z \ge z) dz = \int_0^\infty (1 - e^{-Tz^{-\theta}}) dz.$$

Define $u \equiv Tz^{-\theta}$, and calculate the integral using the method of substitution. Similarly as in calculation of price index, we exploit the formula for gamma distribution. From the expression,

$$\int_0^\infty (1 - e^{-Tz^{-\theta}})dz = \frac{T^{\frac{1}{\theta}}}{\theta} \int_0^\infty (1 - e^{-u})u^{1-\frac{1}{\theta}} du,$$

we obtain after integrating by parts and using formula for gamma distribution:

$$-\frac{T^{\frac{1}{\theta}}}{\theta}\theta(1 - e^{-u})u^{\frac{-1}{\theta}}|_0^\infty + T^{\frac{1}{\theta}} \int e^{-u}u^{\frac{-1}{\theta}} du = T^{\frac{1}{\theta}}\Gamma(1 - \frac{1}{\theta}).$$

Exercise 26 *Prove that $\mathcal{G}_{ni}(p) = G_n(p)$ and derive π_{ni}.*

Solution 27 *Note that the probability that the country i is the supplier of a given good j to country n is*

$$\pi_{ni} = \Pr[P_{ni}(j) \leq \min\{P_{ns}(j); s \neq i\}].$$

The closed form solution is

$$\begin{aligned}
\pi_{ni} &= \Pr[P_{ni}(j) \leq \min\{P_{ns}(j); s \neq i\}] = \int_0^\infty \Pi_{s\neq i}[1 - G_{ns}(p)]dG_{ni}(p) \\
&= \int_0^\infty e^{-[\Phi_n - T_i(c_i d_{ni})^{-\theta}]p^\theta} dG_{ni}(p) \\
&= \int_0^\infty T_i\left(c_i d_{ni}\right)^{-\theta} p^{\theta-1} e^{-T_i(c_i d_{ni})^{-\theta}p^\theta} e^{-[\Phi_n - T_i(c_i d_{ni})^{-\theta}]p^\theta} dp \\
&= \int_0^\infty T_i\left(c_i d_{ni}\right)^{-\theta} p^{\theta-1} e^{-\Phi_n p^\theta} dp \\
&= \frac{T_i\left(c_i d_{ni}\right)^{-\theta}}{\Phi_n} \int_0^\infty \Phi_n p^{\theta-1} e^{-\Phi_n p^\theta} dp \\
(*) \quad &= \frac{T_i\left(c_i d_{ni}\right)^{-\theta}}{\Phi_n}[-e^{-\Phi_n p^\theta}]_0^\infty = \frac{T_i\left(c_i d_{ni}\right)^{-\theta}}{\Phi_n}
\end{aligned}$$

Using (*), we can show that the conditional distribution $\mathcal{G}_{ni}(p)$ of prices of goods actually purchased from source i, defined by

$$\pi_{ni}\mathcal{G}_{ni}(p) \equiv \int_0^p \Pi_{s\neq i}[1 - G_{ns}(q)]dG_{ni}(q),$$

is $\mathcal{G}_{ni}(p) = G_n(p)$. Need to write the integral explicitely, calcel out the terms (*), and note that what we get is G_n.

Exercise 28 *Assume that there is a competitive sector that produces non-tradable service goods using the following production function:*

$$y = Al^{\beta}(\int_0^{\infty} q(p)^{\frac{\sigma-1}{\sigma}} dG_n(p))^{\frac{(1-\beta)\sigma}{\sigma-1}},$$

and assume that the utility function of the household is Cobb-Douglas in tradable and non-tradable components, i.e.:

$$U_n = C_n^{\alpha}[\int_0^{\infty} c_n(p)^{\frac{\sigma-1}{\sigma}} dG_n(p)]^{\frac{(1-\alpha)\sigma}{\sigma-1}},$$

where α is the share of non-tradable goods, and C_n is consumption of the non-tradable good in country n. Furthermore, assume that labor is perfectly mobile across the two sectors producing tradable and non-tradable good. Under this modification, derive the modified labor market clearing condition. HINT: The formula is in the Eaton and Kortum paper. You are asked to to derive it.

Solution 29 *This is Cobb-Douglas function, so the consumer spends α fraction on nontradable goods. As in text, it is still true that fraction β of the expenditures of the entire world on home goods is equal to the total compensation of labor producing these goods (=compensation of labor in country i), and $(1-\beta)$ fraction is equal to payments to intermediate goods:*

$$(1-\alpha)w_i L_i = \beta \sum_n (\frac{X_{ni}}{X_n}) X_n,$$

where X_n denotes total expenditures on tradable goods of a country. Total expenditures of a country on tradable goods are thus given by

$$X_n = (1-\alpha)w_i L_i + (1-\beta)\sum_n (\frac{X_{ni}}{X_n}) X_n,$$

Combining with the abovve, we obtain

$$X_n = (1-\alpha)w_i L_i + (1-\alpha)\frac{(1-\beta)}{\beta} w_i L_i,$$

Furthermore, following the same steps as in the notes, we have

$$\begin{aligned} X_n &= (1-\alpha)w_i L_i + (1-\alpha)\frac{(1-\beta)}{\beta} w_i L_i \\ &= \frac{(1-\alpha)w_i L_i}{\beta}. \end{aligned}$$

Finally, using the above, we derive:

$$\begin{aligned} L_i &\equiv \frac{w_i L_i}{w_i} = \frac{\frac{1-\alpha}{\beta}\sum_n (\frac{X_{ni}}{X_n}) X_n}{w_i} = \frac{\frac{1-\alpha}{\beta}\sum_n (\frac{X_{ni}}{X_n}) \frac{\beta}{1-\alpha} w_n L_n}{w_i} \\ &= \frac{\sum_n \pi_{ni} w_n L_n}{w_i}. \end{aligned}$$

Exercise 30 *Formally derive the above expression for* $\omega(a)$.

Summary 31 *Need to solve for the following probability*

$$\begin{aligned}
\Pr(\frac{Z_2}{Z_1} &\leq a) = \Pr(Z_2 \leq aZ_1) = \int_0^\infty \Pr(Z_2 \leq az)dF(z) = \\
&= \int_0^\infty e^{-a^{-\theta}z^{-\theta}}\theta z^{-\theta-1}e^{-z^{-\theta}}dz = \int_0^\infty e^{(1+a^{-\theta})z^{-\theta}}\theta z^{-\theta-1}dz = \\
&= -\frac{1}{1+a^{-\theta}}\int_0^\infty -e^{(1+a^{-\theta})z^{-\theta}}(1+a^{-\theta})\theta z^{-\theta-1}dz
\end{aligned}$$

Now, need to note that under the integral we have the derivative of $e^{(1+a^{-\theta})z^{-\theta}}$, *and the answer follows.*

Exercise 32 *By integrating over the price schedule, derive the expression for* $\frac{X_{12}}{X_1}$ *calculated below. (NOTE: You are not allowed to use the gravity equation.)*

$$\frac{X_{12}}{X_1} = \frac{1}{P^{1-\sigma}}(\int_{\frac{1}{1+d^{-\theta}}}^1 p_{12}(\omega)^{1-\sigma}d\omega) = \frac{d^{-\theta}}{1+d^{-\theta}}.$$

Solution 33 *Simple algebra gives the solution.*

Exercise 34 *Derive the expression:* $\frac{X_{12}}{X_1} = \frac{\Delta d^{1-\sigma}}{1+\Delta d^{1-\sigma}}$.

Solution 35

$$\begin{aligned}
P_1 &= \left[\int_0^{\frac{1}{2}} p_{11}(\omega)^{1-\sigma}d\omega + \int_{\frac{1}{2}}^1 p_{21}(\omega)^{1-\sigma}d\omega\right]^{\frac{1}{1-\sigma}} = \\
&= \left[\int_0^{\frac{1}{2}} \left(\frac{\omega^{\frac{1}{\theta}}}{A}\right)^{1-\sigma} d\omega + \int_{\frac{1}{2}}^1 \left(d\frac{(1-\omega)^{\frac{1}{\theta}}}{A}\right)^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}} = \\
&= [(*) + (**)]^{\frac{1}{1-\sigma}} = \\
&= \frac{(1-\sigma+\theta)^{\frac{1}{\sigma-1}}}{A\theta^{\frac{1}{\sigma-1}}}\left[\frac{1}{2}^{\frac{1-\sigma+\theta}{\theta}} + \Delta d^{1-\sigma}\frac{1}{2}^{\frac{1-\sigma+\theta}{\theta}}\right]^{\frac{1}{1-\sigma}} = \\
&= \frac{1}{2}^{\frac{1-\sigma+\theta}{\theta(1-\sigma)}} A^{-1}(\frac{\theta}{1-\sigma+\theta})^{\frac{1}{1-\sigma}}(1+\Delta d^{1-\sigma})^{\frac{1}{1-\sigma}}
\end{aligned}$$

where

$$
\begin{aligned}
(*) &= \int_0^{\frac{1}{2}} \left(\frac{\omega^{\frac{1}{\theta}}}{A} \right)^{1-\sigma} d\omega = \\
&= A^{\sigma-1} \int_0^{\frac{1}{2}} \omega^{\frac{1-\sigma}{\theta}} d\omega = \\
&= A^{\sigma-1} [\frac{\theta}{1-\sigma+\theta} \omega^{\frac{1-\sigma+\theta}{\theta}}]_0^{\frac{1}{2}} = \\
&= \frac{A^{\sigma-1}\theta}{1-\sigma+\theta} (\frac{1}{2})^{\frac{1-\sigma+\theta}{\theta}}
\end{aligned}
$$

$$
\begin{aligned}
(**) &= \int_{\frac{1}{2}}^{1} \left(\Delta d \frac{(1-\omega)^{\frac{1}{\theta}}}{A} \right)^{1-\sigma} d\omega = \\
&= A^{\sigma-1} \Delta d^{1-\sigma} \int_{\frac{1}{2}}^{1} (1-\omega)^{\frac{1-\sigma}{\theta}} d\omega = \\
&= A^{\sigma-1} \Delta d^{1-\sigma} [-\frac{\theta}{1-\sigma+\theta} (1-\omega)^{\frac{1-\sigma+\theta}{\theta}}]_{\frac{1}{2}}^{1} = \\
&= \frac{A^{\sigma-1} \Delta d^{1-\sigma} \theta}{1-\sigma+\theta} \frac{1}{2}^{\frac{1-\sigma+\theta}{\theta}}
\end{aligned}
$$

So,

$$
\begin{aligned}
\frac{X_{12}}{X_1} &= \frac{1}{P^{1-\sigma}} (\int_{\frac{1}{2}}^{1} p_{21}(\omega)^{1-\sigma} d\omega) = \\
&= \frac{\Delta d^{1-\sigma}}{(AP)^{1-\sigma}} (\int_{\frac{1}{2}}^{1} (1-\omega)^{\frac{1-\sigma}{\theta}} d\omega) = \\
&= \frac{\Delta d^{1-\sigma}}{(AP)^{1-\sigma}} \left[-\frac{\theta}{1-\sigma+\theta} (1-\omega)^{\frac{1-\sigma+\theta}{\theta}} \right]_{\frac{1}{2}}^{1} = \\
&= \frac{\Delta d^{1-\sigma}}{(AP)^{1-\sigma}} \frac{\theta}{1-\sigma+\theta} \frac{1}{2}^{\frac{1-\sigma+\theta}{\theta}} = \\
&= \frac{1}{2}^{\frac{1-\sigma+\theta}{\theta(1-\sigma)}} A^{-1} (\frac{\theta}{1-\sigma+\theta})^{\frac{1}{1-\sigma}} (1+\Delta d^{1-\sigma})^{\frac{1}{1-\sigma}} \\
&= \frac{\Delta d^{1-\sigma}}{\frac{1}{2}^{\frac{1-\sigma+\theta}{\theta}} (\frac{\theta}{1-\sigma+\theta})(1+\Delta d^{1-\sigma})} \frac{\theta}{1-\sigma+\theta} \frac{1}{2}^{\frac{1-\sigma+\theta}{\theta}} = \\
&= \frac{\Delta d^{1-\sigma}}{1+\Delta d^{1-\sigma}}
\end{aligned}
$$