

Econ 871: Solving DSGE Models Using Perturbation Method

Lukasz Drozd

References

- Uribe, Schmitt-Grohe (2004), “Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function,” *Journal of Economic Dynamics and Control*, vol. 28, January 2004, pp. 755-775
- Judd (1998), “Numerical Methods in Economics”, chapter 13-14, MIT Press
- Klein and Gomme (2006), “Second-order approximation of dynamic models without the use of tensors”, University of Western Ontario manuscript.

- Klein (2000), “Using the generalized Schur form to solve a multivariate linear rational expectations model”, *Journal of Economic Dynamics and Control* 24(10), September 2000, pages 1405-1423.
- Uhlig (1997), “A Toolkit for Analyzing Dynamic Stochastic Models Easily”, <http://www2.wiwi.hu-berlin.de/institute/wpol/html/toolkit.htm>
- Aruoba, Rubio-Ramirez, Villaverde (2006), “Comparing Solution Methods for Dynamic Equilibrium Economies,” *Journal of Economic Dynamics & Control* 30 (2006) 2509–2531.

Packages That Can Do it For You

- Dynare (Platform: standalone or Matlab)
 - <http://www.cepremap.cnrs.fr/dynare/>
- Uribe and Schmitt-Grohe (2004) (Platform: Matlab)
 - http://www.econ.duke.edu/%7Euribe/2nd_order
- Eric Swanson (Platform: Mathematica)
 - <http://www.ericswanson.us/perturbation.html>

Main Idea

- Find a case you know how to solve
- Rewrite the original problem as a parameterized perturbation from this case
- Use Taylor approximation w.r.t. to the perturbation parameter to get an approximate solution
- Verify the accurateness!

Strengths

- By far the best 'local' method: simple, flexible and very fast...
 - Excellent packages available online
 - Possible to take n -th order approximation that makes this method applicable to problems that require at least second order precision (example: portfolio choice problems, see Heathcote and Perri (2007) or Wincoop (2008))

How Does It Work?

- Suppose we want to find the *lowest* value of x that satisfies the equation

$$x^3 - 4x + 0.1 = 0.$$

- It is a cubic equation, and suppose we do not know how to solve...
- To approximate the solution, we will use the fact that we do know how to solve

$$x^3 - 4x = 0.$$

- Factoring x out, we obtain 3 solutions: -2,0,2, the *lowest* solution is -2

Implementing the Perturbation Method

- Step 1: We parameterize the original problem as a perturbation from the case we know how to solve

$$g(\varepsilon)^3 - 4g(\varepsilon) + \varepsilon \equiv 0, \text{ all } \varepsilon$$

where ε is a perturbation parameter, and $g(\varepsilon)$ is the function that returns the lowest solution.

- $\varepsilon = 0$ corresponds to the case we know how to solve
- $\varepsilon = 0.1$ corresponds to our original problem

- Step 2. Using Taylor's Theorem, approximate $g(\cdot)$ by a polynomial

$$g(\varepsilon) \simeq g(0) + g'(0)\varepsilon + \frac{1}{2!}g''(0)\varepsilon^2.$$

Caveats

The Taylor polynomial always exists, providing f is suitably differentiable. But it need not be useful. Consider the example:

$$f(x) = \begin{cases} \exp(-1/x^2) & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

The interest is in f is at 0. It turns out that

$$f(0) = f'(0) = f''(0) = \dots = f^{(n)}(0) = \dots = 0.$$

So, the Taylor polynomial of degree n for f around 0 is

$$P_n(x) = 0 + 0x + 0x^2 + \dots + 0x^n = 0,$$

and so for every n , the residual is $f(x)$.

Clearly in this case, P_n tells us nothing useful about the function.

Fortunately, the family of smooth functions for which Taylor approximation works, called analytic functions, is quite broad. Most simple functions are analytic, and the family is closed under sums, products and compositions. So, as long as the equation we study is an analytic function, the implicit function $g(\varepsilon)$ should be analytic as well. (In the above example it is the improper reciprocal that creates a problem.)

- Step 3. Find $g(0)$, $g'(0)$, $g''(0)$ using the Implicit Function Theorem
 - Since we know that our equation holds identically for all ε , in particular, the first and second derivatives at $\varepsilon = 0$ must be zero. Thus,

$$\begin{aligned}g(0)^3 - 4g(0) &= 0, \\3g(0)^2g'(0) - 4g'(0) + 1 &= 0, \\6g(0)g'(0)^2 + 3g''(0)g(0)^2 - 4g''(0) &= 0.\end{aligned}$$

- From *first* equation we obtain

$$g(0) = -2,$$

- From *second* equation we obtain

$$g'(0) = -\frac{1}{8},$$

- and from *third* we obtain

$$g''(0) = \frac{3}{128}.$$

(Note that $g(0)$ is solved and is then used to solve for $g'(0)$, and both $g(0)$ and $g'(0)$ are used to solve for $g''(0)$. This is a general property.)

Trust, But Verify

- Plugging in to our Taylor expansion, we obtain the approximate solution

$$g(0.1) \simeq -2 - \frac{1}{8}0.1 + \frac{1}{2128}0.1^2 = -2.01246,$$

- It is always a good idea to verify the solution by evaluating the residuals:

$$(-2.0134)^3 - 4(-2.0134) + 0.1 = -0.00062.$$

- As we can see, it is pretty close to zero

Higher Order Approximations

- In principle, we could go as far as we want – we could take 3rd order approximation, 4th order, etc...
- This is the strength of this method
 - Need a more precise solution? Take a higher order approximation

Solving a Simple RBC Model

- As an example, we will now solve a simple RBC model

$$\max E_0 \sum_t^{\infty} \beta^t \log(c_t)$$

subject to

$$c_t + k_t = e^{z_t} k_{t-1}^{\alpha},$$

$$z_t = \rho z_{t-1} + \sigma \varepsilon_t, \quad \varepsilon_t \sim N(0, 1).$$

A Note on Notation

- Note that in the formulation of the model k is shifted to period $t - 1$. This way all t period variables denote variables that are known at period t but not at period $t - 1$
- You will often encounter such 'shifted' notation when an expectation operator is involved

Equilibrium Conditions

- The equilibrium conditions of our model are given by

$$\begin{aligned}c_t + k_t &= e^{z_t} k_{t-1}^\alpha, \\ \frac{1}{c_t} &= \beta E_t \frac{\alpha e^{\rho z_t + \sigma \varepsilon_{t+1}} k_t^{\alpha-1}}{c_{t+1}}, \\ z_t &= \rho z_{t-1} + \sigma \varepsilon_t, \quad \varepsilon_t \sim N(0, 1)\end{aligned}$$

which we can compactly write as

$$\frac{1}{e^{z_t} k_{t-1}^\alpha - k_t} - \beta E_t \frac{\alpha e^{\rho z_t + \sigma \varepsilon_{t+1}} k_t^{\alpha-1}}{e^{\rho z_t + \sigma \varepsilon_{t+1}} k_t^\alpha - k_{t+1}} = 0.$$

What Are We Looking For?

- This model has a recursive representation (SLP chapter 4), and so we know that we are looking for a policy function

$$k(k, z; \sigma)$$

such that for all k, z, σ

$$\frac{1}{e^z k^\alpha - k(k, z; \sigma)} - \beta E \frac{\alpha e^{\rho z + \sigma \varepsilon} k^{\alpha-1}}{e^{\rho z + \sigma \varepsilon} k^\alpha - k(k(k, z; \sigma), \rho z + \sigma \varepsilon; \sigma)} = 0.$$

(The expectation operator E is the integral over ε , which is normally distributed with variance 1 and mean 0.)

- Our goal is to approximate this function

Closed Form Solution?

- Turns out, this particular model has a closed form solution of the form

$$k(k, z; \sigma) = \alpha\beta e^z k^\alpha,$$

where z follows AR(1) process $z_t = \rho z_{t-1} + \sigma \varepsilon_t$, $\varepsilon_t \sim N(0, 1)$.

- Note that the sample paths of the key variables oscillate around deterministic steady state

$$\bar{k} = (\alpha\beta)^{\frac{1}{1-\alpha}},$$

$$\bar{z} = 0$$

- Gives us opportunity to test and better understand the method

Implementing Perturbation Method

- To implement the perturbation method we need to find a case we know how to solve
- We know how to solve for the deterministic case, which corresponds to $\sigma = 0$
 - The solution is given by

$$\begin{aligned}\bar{k} &= (\alpha\beta)^{\frac{1}{1-\alpha}}, \\ \bar{z} &= 0.\end{aligned}$$

- This property makes σ the natural candidate for the perturbation parameter

The Approximation Step

- The second step is to use Taylor expansion to approximate the solution from the known one
- To find out what we are looking for, we first take the Taylor expansion of the policy function

$$\begin{aligned} k(k, z, \sigma) \simeq & k(\bar{k}, 0; 0) + k_k(k - \bar{k}) + k_z z + k_\sigma \sigma + \\ & + \frac{1}{2} k_{kk}(k - \bar{k})^2 + \frac{1}{2} k_{zz} z^2 + \frac{1}{2} k_{\sigma\sigma} \sigma^2 + \\ & + k_{kz}(k - \bar{k})z + k_{k\sigma}(k - \bar{k})\sigma + k_{z\sigma} z\sigma, \end{aligned}$$

where all derivatives are evaluated at the perturbation point $(\bar{k}, 0; 0)$.

What Do We Want?

- From Taylor's expansion, we note that
 - First order approximation requires 4 numbers: $k(\bar{k}, 0; 0)$, k_k , k_z , k_σ
 - Second order approximation additionally requires 6 more numbers:
 k_{kk} , $k_{\sigma\sigma}$, k_{zz} , $k_{z\sigma}$, k_{zk} , $k_{k\sigma}$
- Using equilibrium conditions and the Implicit Function Theorem, our task is to find these numbers
 - The supporting Matlab code for this part can be download from my website

Compact Notation

- To simplify notation, let's define

$$F(k, z; \sigma) \equiv H(k(k(k, z; \sigma), \rho z + \sigma \varepsilon; \sigma), k(k, z; \sigma), k, z; \sigma) \equiv \frac{1}{e^z k^\alpha - k(k, z; \sigma)} - \beta E \frac{\alpha e^{\rho z + \sigma \varepsilon} k^{\alpha-1}}{e^{\rho z + \sigma \varepsilon} k^\alpha - k(k(k, z; \sigma), \rho z + \sigma \varepsilon; \sigma)}.$$

- This way
 - H_i denotes a partial derivative w.r.t. to the i -th argument of H
 - F_k denotes total derivative of H w.r.t. k

First Order Approximation

- Need to find

$$k(\bar{k}, 0; 0), k_k, k_z, k_\sigma$$

- To obtain first order approximation, by analogy to our earlier case, we use the fact that optimal policy must obey the following 4 equations:

$$(1) : F(\bar{k}, 0; 0) = 0$$

$$(2) : F_k(\bar{k}, 0; 0) = 0$$

$$(3) : F_z(\bar{k}, 0; 0) = 0$$

$$(4) : F_\sigma(\bar{k}, 0; 0) = 0$$

Results

- From (1), we obtain

$$k(\bar{k}, 0; 0) = (\alpha\beta)^{\frac{1}{1-\alpha}}$$

- Plugging in $\alpha = 1/2$, $\rho = .9$ and $\beta = .9$, (1) gives

$$k(\bar{k}, 0; 0) = 0.45^2 = 0.2025$$

- From (2), we obtain

$$H_1 k_k k_k + H_2 k_k + H_3 = 0.$$

- Evaluating H_1, H_2 and H_3 for our choice of parameters, (2) gives

$$-16.325 k_k^2 + 44.440 k_k - 18.139 = 0,$$

and solves to

$$k_k = 0.5,$$

$$k_k = 2.22.$$

- But, which solution should we choose, and why do we get two? When $k_k = 2.22$ the system is explosive – and so on this basis we can clearly reject this solution. In general, we will always get explosive solutions,

and will need to manually reject them. This is because the equilibrium system we have used to solve the model simply allows for such solutions. These solution can only be rejected by referring to the equilibrium conditions that we have omitted: the transversality condition, and the non-negativity conditions... In bizarre models (e.g. externalities), the explosive solutions are the right solutions, and can not be rejected! You can find examples of such cases on Lawrence Christiano's website (see lecture notes to his 'short course', Northwestern University). This will not happen in any of our applications.

- From (3), we obtain

$$H_1 k_k k_z + \rho H_1 k_z + H_2 k_z + H_4 = 0$$

- Evaluating, we have

$$43.17 k_k k_z - 4.3710 = 0,$$

which (after plugging in for $k_k = 0.5$) solves to

$$k_z = 0.2025$$

- From (4), we obtain

$$H_1 k_k k_\sigma + H_2 k_\sigma + H_5 = 0$$

- Evaluating, we obtain $H_5 = 0$. Given there is a unique solution to the model, we thus must have

$$k_\sigma = 0.$$

Properties of 1st Order Approximations

- When formulation of the model involves the term $\sigma\varepsilon$ and $E\varepsilon = 0$, then $H_5 = 0$
 - Implication: the first order approximation of policy function does not depend on σ
- To see why, note that H_5 must take the form Expectation of $\{(\text{derivative of } \sigma\varepsilon \text{ w.r.t. } \sigma) \times (\text{some function evaluated at } \sigma = 0, \text{ thus independent on } \varepsilon \text{ as it enters only through terms } \sigma\varepsilon)\}$. The result is $E\{\varepsilon \times \text{constant}\} = 0$.

Remark

- Variance of the shock only enters through higher order terms (like $k_{\sigma\sigma}$, but not the first order term k_{σ})
- So, when the policy function involves a second moment of the shock, like in portfolio choice, first order approximation not good enough from the get go
- Examples: Heathcote and Perri (2007) or Wincoop (2008)

Trust, But Verify

- We obtain the following approximation

$$k(k, z; \sigma) \simeq 0.2025 + 0.5 \times \Delta k + 0.2025 \times \Delta z$$

- Is this any good?
- Can check by taking first order expansion of the true policy at the perturbation point for our choice of parameters

$$k(k, z; \sigma) = \alpha \beta e^z k^\alpha$$

- Let's do it!

- Taking the first order expansion of the true policy function around the perturbation point, we obtain

$$k(k, z; \sigma) \simeq (\alpha\beta)^{\frac{1}{1-\alpha}} + \alpha \times \Delta k + (\alpha\beta)^{\frac{1}{1-\alpha}} \times \Delta z,$$

- Plugging in the parameters, we get

$$k(k, z; \sigma) \simeq 0.2025 + 0.5 \times \Delta k + 0.2025 \times \Delta z$$

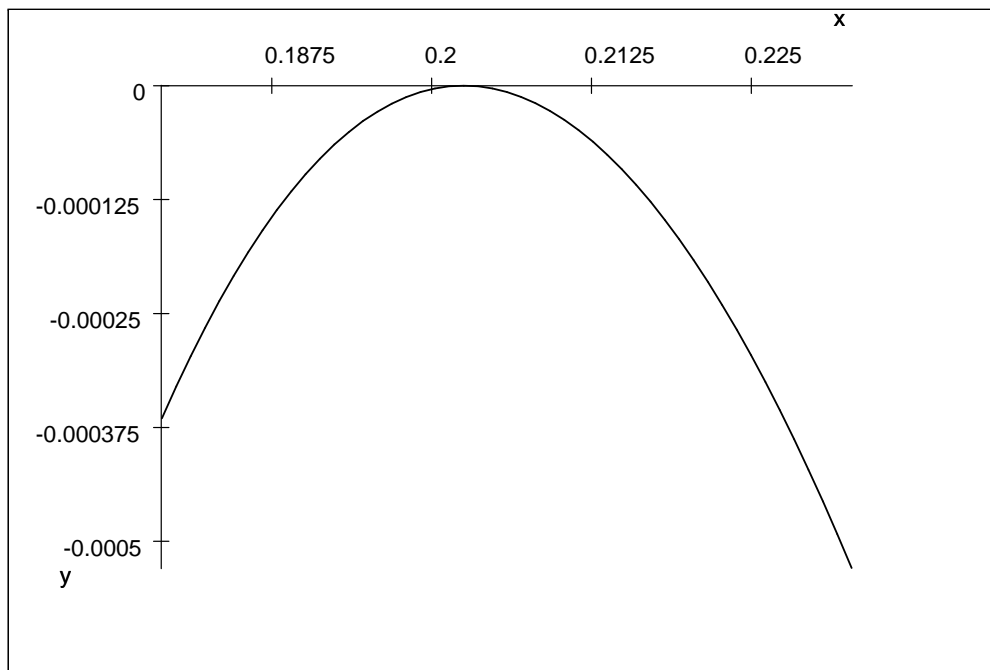
- Works! This is what we got!

How Big Is Approximation Error?

- We approximate $0.45e^z k^{0.5}$ by a linear function, and so the error depends on the curvature of the true policy
- Can plot to see the difference, let's plot the error in k dimension, i.e. let's plot

$$0.45k^{0.5} - [\bar{k} + 0.5(k - \bar{k})]$$

How Good Is This Approximation?



How Good Is This Approximation?

- The error is of order 10^{-3} (when k is within the range $\pm 20\%$ of its steady state value 0.2025)
 - This is more than enough for most applications, but it really depends on the particular application whether it is sufficient or not
- The main advantage of this method is that you can do better whenever you need to

Second Order Approximation

- Need to additionally find

$$k_{kk}, k_{\sigma\sigma}, k_{zz}, k_{z\sigma}, k_{zk}, k_{k\sigma}$$

having

$$k(\bar{k}, 0; 0), k_k, k_z, k_\sigma$$

from the previous step

- To find these numbers we take the second order derivatives of the equilibrium system

- Our second order system is given by

$$F_{kk}(\bar{k}, 0; 0) = 0$$

$$F_{zz}(\bar{k}, 0; 0) = 0$$

$$F_{\sigma\sigma}(\bar{k}, 0; 0) = 0$$

$$F_{kz}(\bar{k}, 0; 0) = 0$$

$$F_{k\sigma}(\bar{k}, 0; 0) = 0$$

$$F_{z\sigma}(\bar{k}, 0; 0) = 0$$

Properties of 2nd Order Approximations

- *Property 1*: The second order system is always linear (in second order terms)
 - To see why, let's evaluate $F_{kk}(\bar{k}, 0; 0) = 0$ for example,

$$\begin{aligned}F_k(\bar{k}, 0; 0) &= k_k(H_1k_k + H_2) + H_3, \\F_{kk}(\bar{k}, 0; 0) &= k_{kk}(H_1k_k + H_2) + k_k[\\& \quad k_k^2(H_{11}k_k + H_{12}) + k_kH_{13} + \\& \quad k_{kk}H_1 + k_k(H_{21}k_k + H_{22}) + H_{23}] + \\& \quad k_k(H_{31}k_k + H_{32}) + H_{33}\end{aligned}$$

Second order terms $k_{kk}, k_{\sigma\sigma}, k_{zz}, k_{z\sigma}, k_{zk}, k_{k\sigma}$ follow from derivatives of the first order terms, and so the system is necessarily linear. Since we know the first order terms from the previous step, we can now use linear algebra to solve it. This is good news!

- *Property 2*: All cross-terms involving σ are zero, i.e. $k_{z\sigma} = 0, k_{k\sigma} = 0$.
 - Note that these terms come from taking derivatives of $F_\sigma(\bar{k}, 0; 0)$ w.r.t. k, z . We have shown that $F_\sigma(\bar{k}, 0; 0)$ is homogenous of degree 1 w.r.t. k_σ , which implies $k_\sigma = 0$. Also, note that we have shown that H_5 is of the form

$$H_5 = E\{\varepsilon \times g(\cdot)\},$$

where g denotes some arbitrary function in which ε appears only through terms $\sigma\varepsilon$. As a result, the derivative of H_5 w.r.t. any variable other than σ must be 0. Thus, when we take the derivative of

$$F_\sigma(\bar{k}, 0; 0) = H_1 k_k k_\sigma + H_2 k_\sigma + H_5,$$

w.r.t. to any variable other than σ , we always obtain

$$F_{\sigma z}(\bar{k}, 0; 0) = k_{\sigma z}(H_1 k_k + H_2) + k_\sigma(\dots\text{some expression}\dots) = 0,$$

and therefore using $k_\sigma = 0$, we have

$$F_{\sigma z}(\bar{k}, 0; 0) = k_{\sigma z}(H_1 k_k + H_2) = 0.$$

As a result, $k_{\sigma z} = 0$ as long as there is a unique solution to the model. (This is not going to be the case with $k_{\sigma\sigma}$, which will be different from zero in general.)

Second Order Approximation

- Need to find $k_{kk}, k_{\sigma\sigma}, k_{zz}, k_{zk}$ using

$$F_{kk}(\bar{k}, 0; 0) = 0$$

$$F_{zz}(\bar{k}, 0; 0) = 0$$

$$F_{\sigma\sigma}(\bar{k}, 0; 0) = 0$$

$$F_{kz}(\bar{k}, 0; 0) = 0$$

$$\begin{aligned}
F_{kk}(\bar{k}, 0; 0) &= k_{kk}(H_1 k_k + H_2) + k_k[\\
& k_k^2(H_{11} k_k + H_{12}) + k_k H_{13} + \\
& k_{kk} H_1 + k_k(H_{21} k_k + H_{22}) + H_{23}] + \\
& k_k(H_{31} k_k + H_{32}) + H_{33}
\end{aligned}$$

gives k_{kk} .

$$\begin{aligned}
F_{kz}(\bar{k}, 0; 0) &= k_{kz}(H_1 k_k + H_2) + k_k[\\
&k_k(k_z(H_{11} k_k + \rho H_{11} + H_{12}) + H_{14})) + \\
&k_{kz} H_1 + k_z(H_{21} k_k + \rho H_{21} + H_{22}) + H_{24}] + \\
&k_z(H_{31} k_k + \rho H_{31} + H_{32}) + H_{34}
\end{aligned}$$

gives k_{kz} .

$$\begin{aligned}
F_{zz}(\bar{k}, 0; 0) = & k_{zz}(H_1 k_k + \rho H_1 + H_2) + k_z[\\
& k_k((k_z(H_{11} k_k + \rho H_{11} + H_{12}) + H_4) + \\
& k_{kz} H_1 + \\
& \rho(k_z(H_{11} k_k + \rho H_{11} + H_{12}) + H_{14}) + \\
& k_z(H_{21} k_k + \rho H_{21} + H_{22}) + H_{24}] + \\
& k_z(H_{41} k_k + \rho H_{41} + H_{42}) + H_{44}
\end{aligned}$$

gives k_{zz} .

(I cross fingers that there are no mistakes in the above derivatives. The code posted online calculates these derivatives using Maple.)

Multivariate Case?

- If k is a vector, there is a non-trivial complication
 - In the first order step, we no longer get a scalar quadratic equation but a matrix quadratic equation (the analog of equation 2)
- Need to know how to solve a matrix quadratic equation

Example: A Multivariate Case

- Let's say we want to solve a model in which there are 2 endogenous state variables (size of k vector), and 1 exogenous state variables (size of z vector), and the equilibrium system is given as a vector H (this time of size 2) s.t. for all k, z, σ

$$F(k, z; \sigma) = H(k(k(k, z; \sigma), \rho z + \varepsilon; \sigma), k(k, z; \sigma), k, z; \sigma) \equiv 0.$$

- Notation: Assume the first 2 arguments of H are $k(k(k, z; \sigma), \rho z + \varepsilon; \sigma)$ arguments, the next 2 arguments are $k(k, z; \sigma)$ arguments, and so on...

Example: A Multivariate Case

- The multivariate analog to our earlier equation (2) (the derivative w.r.t. the endogenous state variables),

$$H_1 k_k k_k + H_2 k_k + H_3 = 0,$$

is now obtained by differentiating

$$\begin{aligned} H^1(k^1[k^1(k^1, k^2), k^2(k^1, k^2), \dots], k^2[k^1(k^1, k^2); k^2(k^1, k^2), \dots], \dots) &= 0, \\ H^2(k^1[k^1(k^1, k^2), k^2(k^1, k^2), \dots], k^2[k^1(k^1, k^2), k^2(k^1, k^2), \dots], \dots) &= 0, \end{aligned}$$

which results in the following multivariate analog of our 'quadratic' term ' $H_1 k_k k_k$ ' ...

$$\begin{aligned}
[H_1^1, H_2^1] & \begin{bmatrix} k_1^1 k_1^1 + k_2^1 k_1^2 \\ k_1^2 k_1^1 + k_2^2 k_1^2 \end{bmatrix} + \dots = 0 \\
[H_1^1, H_2^1] & \begin{bmatrix} k_1^1 k_2^1 + k_2^1 k_2^2 \\ k_1^2 k_2^1 + k_2^2 k_2^2 \end{bmatrix} + \dots = 0 \\
[H_1^2, H_2^2] & \begin{bmatrix} k_1^1 k_2^1 + k_2^1 k_2^2 \\ k_1^2 k_2^1 + k_2^2 k_2^2 \end{bmatrix} + \dots = 0 \\
[H_1^2, H_2^2] & \begin{bmatrix} k_1^1 k_2^1 + k_2^1 k_2^2 \\ k_1^2 k_2^1 + k_2^2 k_2^2 \end{bmatrix} + \dots = 0
\end{aligned}$$

Which in matrix notation, we write as

$$\begin{bmatrix} H_1^1 & H_2^1 \\ H_1^2 & H_2^2 \end{bmatrix} \begin{bmatrix} k_1^1 & k_2^1 \\ k_1^2 & k_2^2 \end{bmatrix} \begin{bmatrix} k_1^1 & k_2^1 \\ k_1^2 & k_2^2 \end{bmatrix} + \dots = \begin{bmatrix} H_1^1 & H_2^1 \\ H_1^2 & H_2^2 \end{bmatrix} \begin{bmatrix} k_1^1 & k_2^1 \\ k_1^2 & k_2^2 \end{bmatrix}^2 + \dots = 0$$

Solving a Multivariate Case

- All equations from the first order system (multivariate analog of equation 2), can be represented as a quadratic matrix equation

$$\psi P^2 - \Gamma P - \Theta = 0,$$

- By Taylor approximation of the policy function, the approximate solution is given by

$$k(k, z; \sigma) = \bar{k} + P \times \Delta k + Q \times \Delta z.$$

- The non-explosive solution requires matrix P to be stable (P^n dies out as $n \rightarrow \infty \dots$)
- *Our task:* Solve for stable matrix P

Solving for Stable Matrix P

- Let m be the number of endogenous state variables in the model, and let's assume that the state space is of minimal size (there are no redundant state variables).
- Our task is to solve the quadratic matrix equation of the form

$$\psi P^2 - \Gamma P - \Theta = 0,$$

for the $m \times m$ matrix P , given $m \times m$ matrices ψ, Γ, Θ .

Theorem

- Define $2m \times 2m$ matrices Ξ and Δ via

$$\Xi = \begin{bmatrix} \Gamma & \Theta \\ I_m & 0_{m,m} \end{bmatrix}, \Delta = \begin{bmatrix} \psi & 0_{m,m} \\ 0_{m,m} & I_m \end{bmatrix},$$

where I_m is the identity matrix of size m , and where $0_{m,m}$ is the $m \times m$ matrix with only zero entries.

- Obtain $2m$ generalized eigenvalues and eigenvectors s of matrix Ξ w.r.t. matrix Δ which, by definition, are the solution to the following equation

$$\lambda \Delta s = \Xi s.$$

Matlab command for finding the generalized eigenvalues and eigenvectors is $\text{eig}(\Xi, \Delta)$, check Matlab help. Select m eigenvalues $\lambda_1 \dots \lambda_m$

that are stable (i.e. satisfy the condition $|\lambda_i| < 1$), and m corresponding eigenvectors $s_1 \dots s_m$ to these eigenvalues. If the model is stable, and the state space is reduced to the minimal size, there will be exactly m non-zero eigenvalues. The eigenvectors will take the form $s_i = [\lambda_i x_i, x_i]$, for some $x_i \in \mathbb{R}^m$, and

$$P = \Omega \Lambda \Omega^{-1},$$

is the stable solution to the matrix equation, where $\Omega = [x_1, \dots, x_m]$ and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$.

Proof

First, examine the last m rows of equation $\lambda\Delta s = \Xi s$. Notice that because of the special form of Ξ, Δ the eigenvectors indeed must be of the form $s_i = [\lambda_i x_i, x_i]$, for some $x_i \in R^m$. Using the first m rows from $\lambda\Delta s = \Xi s$ and plugging in for Ξ, Δ , we obtain

$$\psi x_i \lambda_i^2 - \Gamma x \lambda_i - \Theta x = 0,$$

and thus in matrix form

$$\psi \Omega \Lambda^2 - \Gamma \Omega \Lambda - \Theta \Omega = 0.$$

Multiplying the above by Ω^{-1} from the right, we obtain

$$\psi \Omega \Lambda^2 \Omega^{-1} - \Gamma \Omega \Lambda \Omega^{-1} - \Theta = 0.$$

Noting that

$$\Omega\Lambda^2\Omega^{-1} \equiv \Omega\Lambda\Omega\Omega^{-1}\Lambda\Omega^{-1} = P^2,$$

we have

$$\psi P^2 - \Gamma P - \Theta = 0.$$

Is P a stable matrix? Is it unique? The answer to both questions is yes. The diagonal entries of Λ are all smaller than unity and so P^n can be represented by $P^n = \Omega\Lambda^n\Omega^{-1}$, thus $P^n \rightarrow_{n \rightarrow \infty} 0$. Clearly, if any of the diagonal entries in Λ was bigger or equal to 1, this is no longer true. (See Uhlig (1997) for more details.)

In Practice

- Use the package by Schmitt-Grohe and Uribe (2004) or DYNARE
- The first package is much better for higher order approximations because it takes analytic derivatives
- Except for first order approximation, do not take numerical derivatives! This is what Dynare++ does, and it crashes very often. You really need analytic derivatives.
- In Fortran, compute eigenvalues using the code prepared by Paul Klein (Schur decomposition, solab.f90). See his website and Klein and

Gomme (2006) for codes. BTW, need to paste analytic derivatives from Matlab to Fortran. Matlab has a special function that converts syntax to Fortran syntax. Use it.

- Refer to SLP chapter 6 for theoretical results regarding stability of linear systems