Credit Cards and the Great Recession: The Collapse of Teasers*

PRELIMINARY AND INCOMPLETE

Lukasz A. Drozd
Federal Reserve Bank of Philadelphia

Michal Kowalik
Federal Reserve Bank of Boston

November 5, 2019

Abstract

The U.S. households deleveraged by paying down or defaulting on a massive amounts of credit card debt during and right after the Great Recession. Here, we link deleveraging on credit cards to the supply-driven collapse in promotional activity in this market from late 2008 onward. We document this phenomenon using account-level supervisory data and develop a new theory that is consistent with the large volume of promotional activity prior to the crisis. Using a calibrated model, we show that a shock that depresses the availability of promotional consistent with the data can account for deleveraging. We demonstrate the relevance of this shock for aggregate consumption demand during this time period.

Keywords: deleveraging, teaser rates, Great Recession, balance transfers, unsecured credit
JEL codes: E21, D91, G20

*Drozd (corresponding author): Federal Reserve Bank of Philadelphia, Ten Independence Mall, Philadelphia, PA 19106 (email: lukaszadrozd@gmail.com). The authors thank Bob Hunt, Urban Jermann and Igor Livshits for insightful comments. We are especially thankful to Ricardo Serrano-Padial for his contributions to this project at its inception and early stages. All remaining errors are ours.

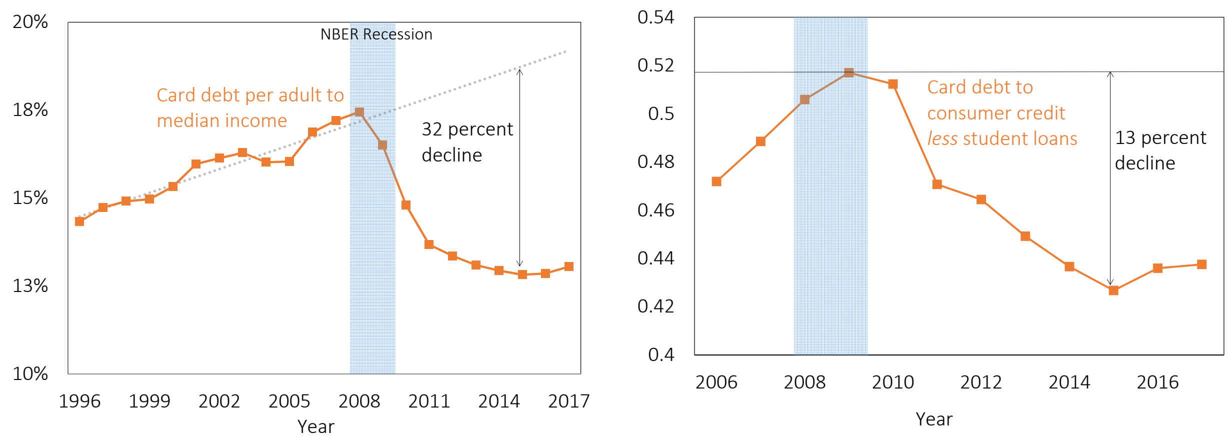

Figure 1: Deleveraging in credit card market.

The figure illustrates deleveraging on credit cards relative to income (median) and relative to aggregate of other consumer credit excl. student loans. Card debt is revolving consumer credit published by the Board of Governors of the Federal Reserve System that mainly comprises of credit card debt, a subcategory of consumer credit that comprises of all consumer debt except debt secured by real estate; median income is median net compensation published by Social Security Administration.

Credit card borrowing was one of the fastest growing consumer debt categories in the late 1980s and throughout the 1990s. By the 2000s, credit cards emerged as a major source of day-to-day borrowing for the U.S. households.[1] The Great Recession brought a sudden stop to this decades-long trend. After a brief run-up of credit card debt early on during the recession, credit card borrowing plummeted relative to the pre-crisis trend (left panel of Figure 1) and fell as a share of total consumer borrowing (right panel of Figure 1). The fact that the turnaround occurred soon after the financial crisis erupted into a major credit crunch within the financial sector begets the obvious question about its relation to the crisis events. In particular, was the decline in credit card borrowing an effect of deteriorating economy or was it an integral part of the transmission channel of the ensuing credit crunch and directly contributed to the recession?

Here we shed light on the answer to this question by linking the 2008-2014 deleveraging to the collapse in promotional activity in the U.S. credit card market – which we document using a unique supervisory dataset covering about half of the U.S. credit card accounts.We develop a new model consistent with the large volume of promotional activity in the credit card market seen prior to the crisis and use it to show that a credit supply shock that depresses the availability of promotional offers as in the data can account for the joint evolution of credit card debt, net chargeoffs, and average

[1]See an excellent monograph by Evans and Schmalensee (2005) documenting the remarkable expansion of credit card lending in the U.S.

interest rate on credit card debt during and right after the Great Recession. This result demonstrates that the credit channel is a plausible explanation for the 2008-2014 deleveraging on credit cards.

The key premise of our analysis is that, prior to the crisis, promotional offering in the credit card market, popularly known as 'teasers,' were ubiquitous and provided a significant discount to a large fraction of credit card borrowers.[2] As we show, about 35 percent of credit card debt had promotional status at the onset of the Great Recession, with zero median interest rate and an average duration of the promotional term of about a year – after which a much higher step-up rate would kick in.[3] Importantly, it appears that borrowers "chained" promotional offers to, in effect, borrow for the long-term at promotional rates. This was made possible by the widely available balance transfer offers that made flipping debt straightforward. Consistent with this view, we find that the volume of promotional balance transfers on the annual basis – underlying the movement of funds to new promotional accounts – was close to the total stock of promotional debt across all account. Much of this activity came to a halt in 2009, around the same time deleveraging on credit cards began. Balance transfers plummeted by more than 70 percent and stayed low for several years. The share of promotional debt bottomed out in 2011 at about half of its pre-crisis level.

To accout for promotional lending, our paper proposes a new theory of credit card lending. The standard theory is not suitable for our analysis because there is no notion of promotional pricing. In fact, as we show, an extension of standard theory that merely allows for promotional offers implies that such offers are largely subotimal and would not arise in a quantitatively meaningful amount. Since the data suggests otherwise, we resort to behavioral modeling inspired by the mico-level evidence documented by Ausubel and Shui (2005) and Agarwal et al. (2015a) on the kind of offers that credit card borrowers receive and choose. This work suggests that credit card borrowers tend to prefer offers on which they end up paying more interest ex post. In particular, using data from a unique experiment, Ausubel and Shui (2005) show that the prevalent choice of promotional offers in the data is at odds with standard geometric preferences and find that naiveté hyperbolic preferences with a discount factor of .81 can rationalize the data. In a similar vain, Agarwal et al. (2015a) show that credit card borrowers prefer low annual fees despite the fact that they pay more in finance charges later. We turn these insights into an equilibrium model of credit card lending with profit maximizing

[2]Our estimate is that the differential between the step-up APR and promotional APR was about 10 percentage points prior to the crisis, even after taking out annualized cost of balance transfer fees that rolling over promotional debt involved.

[3]Among the less risky segments of the market, comprising accounts with 670 or higher FICO scores, the share is 43 percent.

lenders who choose contracts optimally under an incomplete markets setup consistent with the U.S. data and regulations.

In our setup, consumers borrow from a sector of competitive lenders to smooth consumption. Contracts are restricted to unsecured open-ended credit lines that comprise the first period promotional rate, a future step-up rate and a pre-authorized credit limit. Lenders operate under CARD Act of 2009 and cannot raise rates nor cut already utilized credit lines, but they can withdraw unutilized portion of the line, extend it further, and lower the interest rate they charge. Since promotional rates yield a negative profit flow during the promotional period, refinance opportunities in the model arrive with an exogenous delay (possibly random) so that initial lenders break even on promotional offers by charging a higher step-up rate. The key element that drives promotional activity (low promotional rates) is that time-inconsistent consumers overestimate how fast "their future selves" will pay down debt, and hence underestimate the importance of step-up rates.

Our calibrated model predicts a volume of promotional activity and balance transfers consistent with that characterizing prime borrowers in the data (43% of debt has promotional status), which is a little higher as for the sample overall. This is generated by a hyperbolic discount factor 0.8 reported by Ausubel and Shui (2005).[4] By construction, the model is consistent with the following key moments characterizing the U.S. credit market back in 2007: i) 12 months long promotional period on average, ii) an step up rate on promotional accounts of about 17 percent per annum, which yields an average 10 percentage average step-up at expiration of promotional period; iv) net chargeoff rate on credit card loans of 4 percent, iv) gross level of indebtedness of a statistical credit card holder of 21 percent relative to the median income.[5]

To simulate the model, we assume consumers face income shock process that changes during recessionary periods, consistent with the estimated regime switching process by Guvenen et al. (2014). Prior to the crisis, the economy is assumed to be in an expansion for 60 months and the extended recession starts in 2008 and lasts 40 months.[6] To this baseline shock, referred to as the *recession shock*, we add a shock that we refer to as the *collapse of promo shock*. The collapse of promo shock

[4]The number we use is a conservative one. Laibson et al. (2007) estimate a model with a hyperbolic discount factor to fit several patterns that standard models cannot account for. Their estimate is significantly lower. A lower number generally would reinforce our results.

[5]We use Social Security Administration estimate of median net compensation,see. See footnote **??**. This is consistent with the data that we use to estimate income process, as discussed in text below.

[6]The recession is longer than NBER recession but its length is consistent with the extended recessionary dates used by Guvenen et al. (2014) in income process modeling that we adopt and it is also consistent with the decline in median personal income that bottoms out in 2012. Extended recession dates also help match the evolution of the chargeoff rate.

is modeled as an exogenous drop in the probability of receiving a promotional offer with a balance transfer option so as to match the decline in the share of promotional debt in total credit card debt seen in our micro-data between 2008-2016. Since contracts are open-ended, those who do not receive a refinance offer continue to borrow from their incumbent lenders under the CARD Act of 2009 restriction that prevents lenders from hiking rates and slashing credit limits below the level of accumulated debt. The shock is initially unexpected but in the baseline scenario we model it as a permanent regime shift to a lower refinance probability (not essential for results) – consistent with end of the sample levels of promotional activity.

The calibrated model delivers deleveraging matching closely the detrended data. The model generates a hike in the net charge off rate similar to the one in the data[7]. Importantly, due to endogenous selection, the model matches the fact that interest rates actually paid on card debt have not increased dramatically, which is an important check on the internal consistency of our story.[8] Much of the effect on charge-offs comes from the *recession shock*. However, deleveraging almost exclusively comes from the collapse of promotional offerings, as our accounting exercise with the model shows.

Our model suggests a discernible effect of this shock on aggregate consumption demand. According to Mian and Sufi (2014), it was declining consumption demand that was key to the decline in income and employment during the Great Recession. Under this interpretation, an imperfect metric of the aggregate impact is the ratio of consumption to disposable income. In our model, the promo shock results in a decline in this ratio by about two percent, and in the data data, starting from 2008, this ratio fell by five percent. As a caveat to this finding, we do not consider credit substitution. We also leave out about half of the aggregate income concentrated among top income brackets by calibrating the model to match the median income rather than average income, which in the model are roughly equal but not in the data. Credit substitution is unlikely be a major concern due to the declining access to secured credit during this time period. A back of the envelope calculation suggests that the calibration to a median borrower scales down the model implied number by about a factor of two, which is then about a quarter of the number seen in the data.

Related literature. — Our paper contributes to the broader literature studying causal forces

[7]Fraction of debt discharged due to delinquencies that are 180+ days overdue, net of current flow of recoveries. This is aggregate statistic reported by the Board of Governors of the Federal Reserve System.

[8]A simple back of the envelope calculation suggests that, if 35 percent of debt suffered several hundreds basis points hike on average, this increase should have resulted in a several hundred basis points hike in the average rate on credit card debt. Our model shows such a calculation is overstating the facts due to endogenous selection. The interest rate on outstanding debt rises by less than 100 basis point in the model.

behind the Great Recession. Substantively, we document a new fact: the collapse of promotional activity in the U.S. credit card market after 2008, which we show is consistent with observed deleveraging on credit cards and potentially relevant for aggregate demand after late 2008. In this context, the most closely related work is Mian and Sufi (2010), whose findings we complement structurally and substantiate the underlying mechanism. Mian and Sufi (2010) show that the reliance on credit cards across the U.S. counties prior to the crisis has been a strong predictor of the decline in auto sales across later on during the crisis in addition to household leverage.[9] Our paper contributes to modeling of the microstructure of the U.S. credit card market as part of the debate on regulating consumer products. We develop a novel and unique framework that allows to study the role of promotional pricing of credit card debt. We characterize the optimal contracts, and show that the theory can quantitatively match the high volume of promotional activity in the U.S. credit card market prior to the crisis. In terms of mechanism, we build on the agenda attributed to Laibson et al. (2007), as well as that by Ausubel and Shui (2005) in the narrower context of credit cards, whose findings are instrumental to our analysis. We complement this work by studying a complex Ramsey problem of optimal choice of promotional rates in dynamic setting with default risk. In terms of interpretation of our findings, the work by Agarwal et al. (2015b) is particularly relevant. They show that CARD Act of 2009 had little effect on lender behavior, suggesting that factors other than the CARD Act of 2009 played a role in driving the decline in promotional activity that we document.

1 Stylized facts

This section establishes four stylized facts that are key to our analysis: i) Prior to the crisis, an estimated 35 percent of credit card debt had promotional status and 43 percent among prime borrowers with FICO scores 670 or higher; ii) promotional debt provided a major discount relative to the going rates for an average duration of about a year; iii) a large volume of balance transfers sustained the stock promotional debt, consistent with the idea "chaining" of promotional offers to, in effect, borrow for the long-term; iv) the collapse of promotional activity has been orthogonal to changing borrower characteristic and coincident with deleveraging on credit cards in the aftermath of the 2008 financial crisis.

[9] Mian and Sufi (2010) proxy reliance on credit cards by the utilization rate in 2006 and include a broad range of demographic controls as well as a measure of household leverage in each county. A back of the envelope calculation based on the numbers they report on page 20 and in Table 7 (column 1 and 3) suggests that counties that had a higher utilization by one standard deviation in 2006 experienced a decline in auto sales by 1/6 of standard deviation.

1.1 Data sources and descriptions

We use three data sources: 1) aggregate data from the Board of Governors of the Federal Reserve System, such as the stock of revolving debt (consumer credit - G.19) and net charge-off rate on credit cards (charge-off and delinquency rates on loans and leases at commercial banks), downloaded from the Federal Reserve Bank of St. Louis (FRED); 2) supervisory OCC/Y14M account level micro-data focusing on general purpose credit cards from 6 large credit card lenders tracked between 2008 and 2017, and eight in total, having an approximate market share of over 50 percent in 2007 (accounting for 30 percent of general purpose card credit card accounts);[10] 3) Experian credit bureau database comprising of a representative panel of 200,000 credit records tracked between 2001 and 2013.[11]

1.2 Credit card borrowing prior to the crisis

Table 1 characterizes promotional activity in the U.S. credit card market prior to the crisis. We use the first quarter of 2008, since it is the first observation available in our data. To the extend that the phenomena we focus on had already started by that time, our findings would only understate them.

The first panel of the table shows that the ratio of debt on promotional cards to debt on all cards for all borrowers and prime borrower (670+ FICO).[12] As we can see, in the first quarter of 2008 35 percent of debt for all borrowers and 43 percent for debt with FICO score 670+ resided in promotional accounts.

The second panel looks at refinancing. The accounts in our database are not linked and we only see inbound balance transfers to new or existing accounts and we know the promotional status of these accounts (funds). We find average duration of promotional period to be 12 months, surprisingly unconditional promo spell on all accounts rather than those originated in 2008 is longer. This implies that after about 12 months half of promotional debt would had to be renewed as of 2008. We find that the volume of promotional balance transfers closely matches the stock of debt that must be

[10] The data comes from a confidential supervisory collection the Federal Reserve System maintains for the purposes of the Dodd-Frank Act Stress Test (DFAST). The data is on an account level with a monthly frequency and is provided by bank holding companies subject to DFAST. The sample before 2013 is limited to several largest banks and it comes from OCC merged data with Y14M reporting. We focus on this sample here. Data after 2013 covers a broader sample of banks. The dataset is related to the one used by Agarwal et al. (2015b). Appendix A provides a detailed description of variables we use.

[11] The credit bureau data summarizes credit history of 200,000 credit market participants: the first 100,000 records are representative as of 2001 and the second one is representative as of 2013. We use observations from both panels. The database has been purchased from Experian by the author for research purposes using a standard proprietary data purchase agreement. The agreement and contact information are available upon request.

[12] We define debt in month t as the difference between the balance in month $t-1$ net of any payments made by the borrower in month t.

Table 1: Promotional credit card lending prior to the crisis.

Statistic	2008Q1
a. Use of promotional debt:	
Promo debt to total debt[a] [%]	34.8
Promo debt with 670+ FICO to total debt [%]	42.7
Promo debt with at least 50% APR discount to promo debt[b] [%]	67.8
b. Refinancing and balance transfers:	
Median duration of promo spell (originated in 08)[c] [months]	10
Average duration of promo spell (originated in 08)[c] [months]	12
Median duration of promo spell (all accounts)[c] [months]	12
Average duration of promo spell (all accounts) [c] [months]	16
Balance transfers (BT) per annum to promo debt [%]	131
BT to promo cards to BT total [%]	92.11
BT to flow of promo debt nearing expiration (last quarter) [%]	103.75
Average transferred amount per BT	$4,290
c. Interest rates (in APR):	
Median promo APR [%]	3.53
Average promo APR [%]	4.33
Average promo APR with discount 50%+ debt [%]	2.61
Average non-promo APR [%]	15.52
Average step-up APR on promo accounts w/ debt [%]	17.31
Median step-up APR on promo accounts w/ debt [%]	15.99

Notes: The table illustrates the state of the U.S. credit card market prior to the crisis. [a]Debt are credit card balances carried over for at least one billing cycle. Calculating debt requires lagged balance, and hence all numbers for Q1 2008 start from February. [b] Promo debt on low APR is the promo debt for which the promotional APR is lower than the step-up APR by at least 50 percent. [c] The spell is a number of months for which an account has a positive promotional balance, among accounts originated in 2008. We find equal median and higher mean for all accounts, which suggests accounts originated prior to 2008 had a longer promotional spell.

renewed even under our conservative estimate of 12 months long promotional spell. We also compare the flow of balance transfers to the flow of debt that nears the expiration of promo period (average promo debt seen one quarter before the promo flag is removed to quarterly balance transfer volume) and find a similar number.

The last panel shows that promotional borrowers enjoyed a several percentage points discount relative to rates on both non-promotional debt and step-up rates on promotional accounts. Even factoring fees, typically charged on balance transfers, the gap is still sizable. For example, assuming a balance transfer fee of 3 percent, which applies to the majority of transfers.[13], these numbers imply an average discount of about 10 (=17-4-3) percentage points over the average step-up rate over the average duration of promo of 12 months.

[13]As reported by Grodzicki (2015) based on the analysis of Mintel credit card solicitations data.

1.3 The 2008 collapse of promotional activity on credit cards

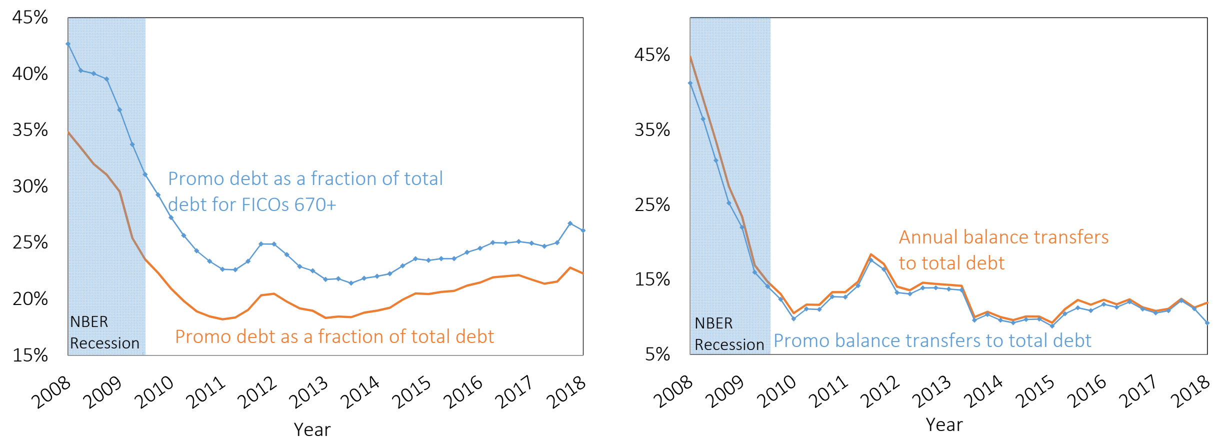

Figure 2: Collapse of promotional activity in the U.S. credit card market.

Notes: The figure illustrates the decline in the share of promotional credit card debt to total debt (left panel) and the collapse of balance transfers (promotional balance transfers) as a fraction of debt. Promotional debt is debt having a promotional status and lower rate. Balance transfers are reported on annual basis. Promotional balance transfers are transfers to promotional accounts. Data after 2013 focuses on OCC original sample of banks.

The 2008 financial crisis and the preceding subprime mortgage crisis were coincident with a persistent collapse in promotional activity in the credit card market. The thus far omnipresent promotional "teaser" offers essentially vanished from consumer's mailboxes, with mail-in solicitations declining by 75 percent between 2007 and 2009.[14] Consequently, as Figure 2 shows, balance transfer plummeted by more than 70 percent and the fraction of promotional debt to total credit card debt steadily declined to about 50 percent of its pre-crisis value by 2011. As Figure 3 shows, the average interest rate on credit card debt rose by about 100 basis points despite declining interest rates in the economy. From the perspective of our model, the evolution of the average interest rate paid on credit card is an important check on the internal consistency of its predictions.

The collapse in promotional activity has been orthogonal to the deteriorating risk characteristics of the borrower pool. This observation suggests that lenders were *not* merely responding to falling credit scores or rising delinquency rates. The decline was broad-based and affected all risk segments of the market.

[14]Interestingly, the response rate to solicited offers was slightly rising despite a declining trend in the pre-crisis period and worse terms being offered. Sources: Data retrieved from Synovate mail monitor in 2011 (www.synovate.com/mailmonitor). Solicitation data we use can be found in "Report to the Congress on the Profitability of Credit Card Operations of Depository Institutions," Board of Governors of the Federal Reserve System, June 2007-2012 (https://www.federalreserve.gov/publications/other-reports/credit-card-profitability-2012-general-discussion.htm).

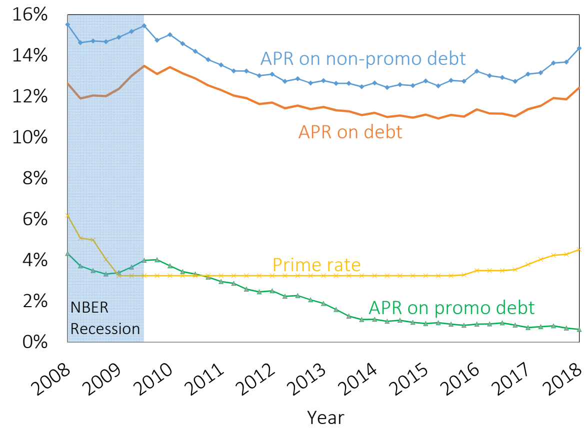

Figure 3: Annual percentage rate (APR) on credit card debt.

Notes: The figure shows average interest rate paid on credit card debt depending on its promotional status, as well as step-up rate on promotional accounts with debt and the prime rate for comparison.

We supplement our account level analysis using credit bureau data. We report the numbers for deleveraging and numbers of balance transfers as a cross-check. This data is biannual and limited in scope, but it is unique in the sense that it contains information on the number of balance transfers executed by each individual within the last 12 months. We cannot distinguish between debt and balance like in our account-level data, and instead we focus on card users (individuals carrying current balances). We consider the entire sample, and to get closer to a median card holder, we trim the extremes and consider individuals with total credit card balances across all accounts above 3,000 dollars and less than 100,000.

Using the data, we first build a linear probability model of balance transfers based on a wide array of variables summarizing riskiness of the borrower, including credit score, delinquencies, total balances, credit card average utilization rate, mortgages on file etc. The model is estimated off sample and covers years prior to the crisis. The full list of variables can be found in Appendix A. We then compare the number of balance transfers per card user to the predicted value by the regression, after aggregating both to obtain a sample average. As Figure 4 shows, the actual number declines by about 65 percent. The decline is largely orthogonal to changing risk characteristics of the borrower pool.

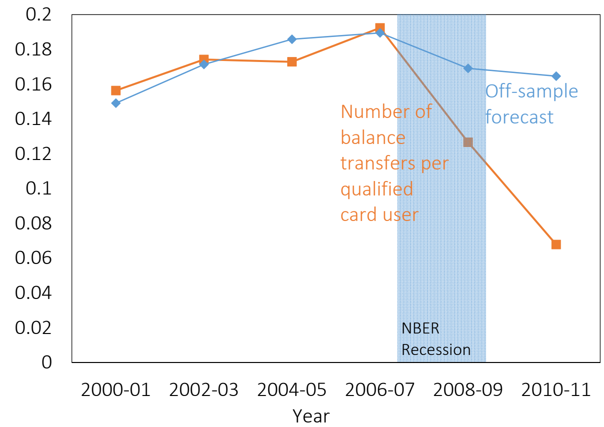

Figure 4: Balance transfers per qualified card user in Bureau Data: predicted versus actual.

Notes: The figure plots the average number of balance transfers per qualified card user over a time period of 12 months versus predicted value using linear probability model based on individual borrower's risk characteristics. See Appendix A for a complete list of variables. A qualified card user carries card balances of a at least 3,000 dollars and less than 100,000 dollars and has a Vantage score of 600 or more (good credit score).

1.4 The 2008-2011 deleveraging on credit cards

Figure 1 (left panel) shows that starting from late 2008 credit card debt per adult to median income fell by about 32 percent relative to trend.[15] Figure 5 shows peak-to-trough evolution of credit card debt held by banks in our OCC/Y14M sample and the gross chargeoff rat on their portfolio, compared to total revolving debt reported by Federal Reserve Board and net chargeoff rate on credit card debt among 100 largest banks. Debt is not readily comparable to aggregate data as it may indicate that the banks in our sample are gaining or losing market share. The decline starts later and it both more persistent and bigger. The evolution of gross chargeoffs is more in line with chargeoff rate among 100 largest banks. See Mian and Sufi (2010) for a complementary cross-sectional evidence, as discussed in related literature.

[15] Median income pertains to *median net compensation* published by Social Security Administration. This measure is approximately gross W2 taxable income. We use median income because it provides a better representation of typical individual in the economy, mainly because our model is not able to capture income skewness present in the data and median income and average income are equal in the model but not in the data. The pattern does not depend on this normalization and it is identical when CPI or disposable income is used instead. It only affects the level, which lower by about 40 percent in case of disposable income.

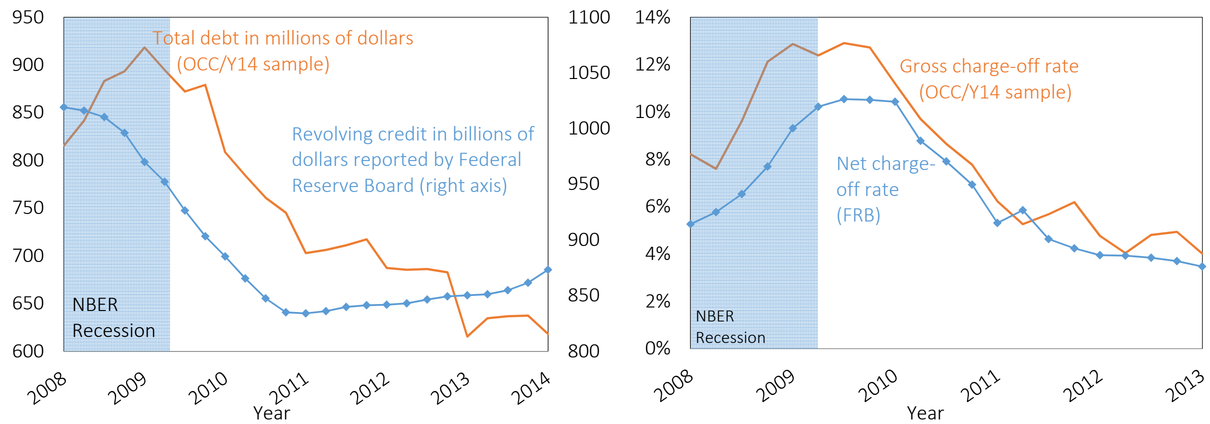

Figure 5: Size and chargeoffs on credit card portfolio held by banks in OCC/Y14M sample.

Notes: The left-panel of the figure shows the decline in debt held by banks from our OCC/Y14M sample relative to total revolving debt reported by Federal Reserve Board, which in part may indicate changing market shares of the banks in our sample. The right-panel shows gross charge-off rate on their credit card portfolio and compares it to net charge-off rate reported by Federal Reserve Board for 100 largest banks. Net charge-off rate is typically lower by 1 to 2 percentage points than gross charge-off rate because the former subtracts collections and recoveries.

2 Mechanism: A three-period prototypical economy

In this section we lay out the centerpiece of our theory: the three-period equilibrium contracting problem that involves open-ended credit lines, promotional offers, refinancing risk and hyperbolic discounting. Our later model embeds this contracting problem into a $T-$period life-cycle environment. We use this model to analytically characterize the mechanism that governs the use of promotional offers, both under geometric discounting and hyperbolic discounting. In particular, we show that absent hyperbolic discounting promotions do not arise. We then show that hyperbolic discounting naturally leads to promotional offers for even modest parameter values. We explain how chaining of promotional offers propagates the use of promotional offers. Finally, we discuss macroeconomic consequences of promotional pricing.

2.1 Environment

The model comprises a continuum of consumers and a large number competitive lenders. Consumers start with some debt and use unsecured credit lines extended by lenders to smooth consumption. Lenders extend open-ended credit lines to consumers. They have unlimited access to funds at no cost but must break even in expectation.

2.1.1 Consumers

Consumer preferences feature naiveté hyperbolic discounting. Accordingly, consumers ex ante evaluate consumption streams according to preference given by

$$u(c_1) + \eta\beta[u(c_2) + \beta u(c_3)], \tag{1}$$

while their ex post choices are *always* determined by

$$u(c_2) + \eta\beta u(c_3), \tag{2}$$

where c_t denotes consumption in period t and $0 < \eta \leq 1$, $\beta > 0$. Clearly, if $\eta = 1$, consumer preferences are standard. We will say that consumers discount the future geometrically in such a case. If $\eta < 1$, consumers are time-inconsistent because they underestimate how much they will borrow in the second period. We will say that consumer discount the future hyperbolically.

Income Y in each period is exogenous and it is determined by a discrete random variable s that here follows a two state Markov process independent across consumers: $Y(s = 0) = \overline{Y}$, $Y(s = 1) = \underline{Y} < \overline{Y}$. Consumers' first period (initial) state is the high state $s = 1$. We assume that the switches to low state with probability p and remains unchanged with probability $1 - p$. For simplicity, we assume that consumers always default in the low state and repay in the high state. Equivalently, the cost of defaulting is assumed extremely convex in income.

2.1.2 Lenders

Lenders compete in a Bertrand fashion. Accordingly, they maximize consumer utility subject to a zero profit condition that must hold in expectation. Credit is restricted to unsecured credit lines and consumers can only have one open credit line at a time. There are no other credit instruments in this economy.

A credit line comprises credit limit $L \geq 0$, a rate F charged on borrowing in the first period and a step-up rate R that places the upper bound on rates thereafter (i.e., lenders can always lower rates below the step up rate but cannot increase it). Step-up rate can be lowered but cannot be raised. The line is open-ended in the sense that lenders cannot cut credit limits on the utilized portion of the credit line to force early repayment. These contractual restrictions are consistent with the Credit Card Accountability Responsibility and Disclosure (CARD) Act of 2009, which partially come into effect in 2009 and fully in 2010. They are arguably also consistent with the prevalent industry practice prior to the enactment of the CARD Act, although the extent may be debatable.[16] We normalize lender cost of funds to zero and the law of large numbers is assumed.

Our goal is to characterize the equilibrium contract extended in the first period. This contract comprises a triple $\mathcal{C} = (F, R, L) \in \Theta$ that solves

$$\max_{(F,R,L)\in\Theta} U(F, R, L) \quad s.t. \quad \Pi(F, R, L) = 0. \tag{3}$$

where U is the indirect utility of the consumer accepting (F, R, L) as her initial contract and Π is the present expected discounted profit of the initial lender who extend this contract. Θ assumes non-negative rates and limits, $F \leq R$, and non-tight upper bounds to ensure compactness. (See equation (24) in the Appendix.)

2.1.3 Lending market microstructure

We now define U and Π by laying out how consumers and lenders interact in the lending market. To simplify the setup, we lay it out by starting from the last period.

Lending in the third period. Assume the high income state $s = 1$ has persisted to the third period. In such a case, the borrower has not defaulted yet, and her relevant state as of the beginning of the third period is her second period debt b_2. If income state is low, the borrower defaults on b_2 and consumes her income $\underline{Y}$. If income state continues to be high, she repays. Accordingly, consumption in the high state in the third period is $c_3(b_2; s = 1) \equiv c_3(b_2) := \overline{Y} - b_2$. The lender is paid back the principal with probability p.

[16] According to CARD Act lenders must maintain interest rate on accounts up to five years, and at most double the minimum payment. In particular, lenders cannot cut credit limits on the utilized portion of the credit line to force early debt repayment. However, lenders can set promotional rates that expire after a specified period of time that is not shorter than 6 months. Before the CARD Act, term changes were possible and did happen. However, anecdotal evidence suggest that banks often recognized the value of reputation. For example, Capital One and Citi, which together account for about thirty percent of the market, were contractually committing themselves to offer opt out options from any rate changes other than the ones triggered by noncompliance (e.g. late payments, overdraft etc...). This was prior to the CARD Act. In early 2008, Chase, followed suit and also adopted an internal rule of not responding to any credit history changes when reviewing terms on credit card accounts. Chase adds another ten percentage points to the market share. To top it off, OCC openly discouraged national banks from the practice of changing terms on credit cards (see OCC Advisory Letter, AL 2004-10). For this and other evidence, see Appendix to H.R. 5244 "The Credit Cardholders' Bill of Rights: Providing New Protections for Consumers," Hearing before the Subcommittee on Financial Institution and Consumer Credit of the Committee on Finance Services, U.S. House of Representatives, One Hundred Tenth Congress Session, April 17, 2008, Serial no. 110-109. Pages 280, 327, 371, 373-379, and 410 are of particular interest. See also the excellent monograph by Evans and Schmalensee (2004). Argus, Information & Advisory Services, LLC, reports that the incidence of rates being priced up doubled between 2008-2011 relative to pre-crisis levels, although the analysis counts all balances and does not distinguish between changes on existing debt, balances and new balances. No significant changes are reported for fees. See files.consumerfinance.gov/f/2011/03/Argus-Presentation.pdf.

Lending in the second period. The borrower's endogenous state is her debt b_1 accumulated in the first period and contract on hand $\mathcal{C}$. If her exogenous income state s switches to low, which occurs with probability p, the borrower defaults and consumes $\underline{Y}$. The initial lender loses the principal b_1.[17] What happens after with the borrower is not relevant, since default severs the relation with the lender and borrower's continuation is independent of her debt because debt is wiped out.

If, on the other hand, the income state continues to be high, which occurs with probability $1-p$, the borrower receives a competitive *market refinance offer* $\mathcal{C}_M^\eta = (F_M^\eta, R_M^\eta, L_M^\eta)$. After the borrower decides whether to accept of reject market offer with no recall, the initial lender (incumbent lender hereafter) reprices her line $\mathcal{C}$ to $\mathcal{C}_I^\eta = (F_I^\eta, R_I^\eta, L_I^\eta)$ under the legal constraint that $\max\{F_I^\eta, R_I^\eta\} \leq R$ and $L_I^\eta \leq b_1$. Given market offer is extended nonstrategically, it is without loss to assume that the incumbent lender's repricing policy is only a function of the ex post value of consumer's decision to refinance $\lambda = 0, 1$ ($\lambda = 1$ means the consumer refinances).

If the borrower refinances ($\lambda = 1$), for a fraction ρ of the second period she still continues with the incumbent lender. This important friction, referred to as the *refinance friction* , is what allows to sustain promotional offers in equilibrium. Without this friction promotional zero profit offers would be ex post undercut and therefore trivially unprofitable ex ante.

The second period market offer must earn interest p, which requires $F^M = p/(1-\rho)$, and it features a slack credit limit L_M. This is because the third period is terminal and there are no interest payments in the third period. Accordingly, if income state is high, consumption in the second period under repayment is

$$c_2(b_1, F_I^\eta(\lambda); b_2, \lambda) := \begin{cases} \overline{Y} - b_1 - (\rho F_I^\eta(\lambda) + (1-\rho)\frac{p}{1-\rho})b_2^+ \text{ if } \lambda = 1 \\ \overline{Y} - b_1 - F_I^\eta(\lambda) b_2^+ \text{ if } \lambda = 0, \end{cases} \tag{4}$$

where b_2^+ is shorthand for $b_2^+ = \min\{0, b_2\}$. The profit of the initial lender from the second- and third-period conditional on the high state persisting throughout is

$$\pi_2(b_2, \lambda; F_I^\eta(\lambda)) := \lambda\rho F_I^\eta(\lambda) b_2^+ + (1-\lambda)(F_I^\eta(\lambda) - p) b_2^+$$

The consumer's choice of b_2 in the second period is constrained by $\min\{L_M^\eta, L_I^\eta\}$, which is non-binding in equilibrium and we can drop it without loss. The market offer relaxes the borrowing constraint in the second period. An analogous reasoning applies to L_I^η.[18] This result is formalized in

[17]Note that our timing assumes that there is enough time to retract the line down to debt b_1 and the borrower default on amount equal to her debt rather than credit limit. This also implies that there is no benefit of having slack credit limits in our setup.

[18]If the borrower refinances, since the incumbent bears no default risk, more borrowing only raises both incumbent's

the lemma below.(All proofs are relegated to Appendix B.)

Lemma 1 *The constraint $b_2 \leq \min\{L_I, L_M\}$ is nonbinding in equilibrium.*

Lending in the first period. Upon receiving her first period contract, $\mathcal{C} = (F, R, L)$, the consumer decides b_1 while forming expectations regarding incumbent lenders' (conditional) repricing policy in the second period. When $\eta = 1$, her expectations are rational and equal to the actual lender policy in equilibrium. But, if $\eta < 1$, the consumer is time-inconsistent and may believe that the lender will reprice in the next period for her "patient" self, which is incorrect ex post and does not occur on the equilibrium path. As an alternative, if consumer type change is not readily observable to lenders, it can be the case that lenders reprice to the population, implying rational expectations. If type never switches, lenders would not use screening contracts and borrowers should assume that. As our baseline, we favor the former case but keep the discussion general. We will say that a borrower is *ex ante rational* if on equilibrium path repricing aligns with her expectations ex ante, which, for any fixed contract, may also happen under the first regime.

Let $\hat{F}_I(\lambda)$ be the consumer's ex ante expectation of what incumbent's ex post rate repricing policy is. As mentioned, we refer to expectations as ex ante rational iff $\hat{F}_I(\lambda) = F_I^\eta$, which in the baseline case requires $F_I^1 = F_I^\eta$. Accordingly, consumption associated with the initial contract $\mathcal{C} = (F, R, L)$ is

$$c_1(b_1) := \overline{Y} - B + b_1 - Fb_1^+. \tag{5}$$

The indirect utility function – which without loss omits terms pertaining to the continuation utility following low income realizations[19] – is

$$U(F, R, L) = \max_{b_1 \leq L, b_2, \lambda} u(c_1(b_1)) + \beta\eta(1-p)\left[u(c_2(b_1, \hat{F}_I; b_2, \lambda)) + \beta(1-p)u(c_3(b_2))\right]. \tag{6}$$

The profit from the first period of the initial lender is $\pi_1(b_1; F) := (F - p)b_1^+$. For later use, we denote the policy functions that solves the ex ante problem by $b_1(F, R, L), b_2(F, R, L), \lambda(F, R, L)$.

If the consumer discounts the future hyperbolically, the policy functions derived from the above problem are not sufficient to calculate profits. This is because the borrower makes a different borrowing decision in the second period and lenders are aware of it. It is clear that ex post the borrower

profits and also borrower's utility (if constraint indeed binds). If, on the other hand, the borrower stays with the incumbent the incumbent's continuation profit is the same as that of a new lender, and hence setting a binding credit limit would only make sense for $F_I < p$, which can only happen if the legal constraint binds, i.e., $R < p$. This, however, cannot arise in equilibrium because initial lender would not break on $R < p$.

[19]Note that continuation utility from paths following low income realizations only shift the direct utility function upwards by a constant and hence are irrelevant for the contracting problem at hand.

should have rational expectations regarding incumbent's repricing policy, which we denote by $\hat{F}_I^{\eta}$ to distinguish it from her ex ante expectations $\hat{F}_I$. Since the last period is terminal, and the step up rate R no longer restricts the choices, the ex post refinancing policy λ^{η} picks the lowest rate, i.e.,

$$\lambda^{\eta}(\hat{F}_I^{\eta}(\lambda=1), F_I^{\eta}(\lambda=0)) := \begin{cases} 0 \text{ if } \hat{F}_I^{\eta}(\lambda=1) \leq \rho F_I^{\eta}(\lambda=0) + p \\ 1 \text{ otherwise.} \end{cases} \tag{7}$$

Consequently, ex post borrowing solves

$$b_2^{\eta}(F_I^{\eta}(\lambda), \lambda) := \arg\max_{b_2} u(c_2(b_1, F_I^{\eta}(\lambda); b_2, \lambda)) + \beta\eta(1-p)u(c_3(b_2)). \tag{8}$$

Given policies $b_1(\cdot), b_2^{\eta}(\cdot), \lambda^{\eta}(\cdot)$ and equilibrium $F_M = p/(1-\rho)$, the ex ante profit of the initial lender is

$$\begin{aligned} \Pi(F, R, L) =& \pi_1(b_1(F, R, L), F) + \\ & (1-p) \max_{F_I(\lambda) \leq R} \pi_2(b_2^{\eta}(F_I(\lambda), \lambda), \lambda^{\eta}(\hat{F}_I^{\eta}(\lambda=1), F_I(\lambda=0)); F_I(\lambda)). \end{aligned} \tag{9}$$

As a general property, for any b_1 chosen ex ante, if b_2 solves (6), we have $b_2 < b_2^{\eta}$ when $\eta < 1$. This is the key wedge that hyperbolic discounting introduces in our model. Finally, since there is no value of having slack credit limits, as ex post lenders always can ex post retract unutilized lines, without a loss we can focus attention on contracts that are fully utilized, i.e., $L = b_1$. This is summarized in the lemma below.

Lemma 2 *Without a loss, contracts equilibrium contract is fully utilized in equilibrium, i.e.,* $L = b_1^+$.

We have now completed the definition of the contracting problem in (3). Equilibrium contract is the fixed point of this problem such that borrower expectations $\hat{F}$ and $\hat{F}^{\eta}$ are aligned with equilibrium policies, i.e., in the baseline case $\hat{F} = F_I^1$ and $\hat{F}^{\eta} = F_I^{\eta}$.

2.2 Characterization of equilibrium contracts

We are now ready to characterize the equilibrium contract. We focus attention on parameter values that result in positive credit usage, i.e., $b_1 > 0$ in equilibrium. $B > 0$ is enough to ensure this is the case, as we show in the Appendix. We assume that the lender's profit function is increasing in F and R around the zero profit flat contract $F = p = R$. In addition, to state our results, we will say that a contract (F, R, L) satisfies **local monotonicity (LM)** when policy functions $b_1(F, R, L)$, $b_2(F, R, L)$ are strictly decreasing in F and R in the neighborhood of (F, R, L). This means that the usual substitution effect locally dominates the income effect. It is clear that this is an empirically relevant case.

2.2.1 Geometric consumers ($\eta = 1$)

We begin with a lemma that applies whenever the consumer is ex ante rational regarding her expectations of incumbent's repricing policy. This is automatic in the case of geometric consumers but not in the case of hyperbolic consumers. (All proofs are relegated to Appendix B.)

Lemma 3 *Without a loss,* $F_I^{\eta=1}(\lambda) = \hat{F}_I = R$.

The lemma follows from the simple fact that extending a contract that is repriced ex post cannot be better then extending the repriced contract right away, given the borrower rationally expects it anyway. Without a loss, we can restrict attention to contracts that are not repriced ex post.

Using the lemma, the indirect utility in (6) – which, recall, omits terms pertaining to the continuation utility following low income low state realizations – simplifies to

$$U(F, R, L) := \max_{b_1 \leq L, b_2} u_1(c_1) + \beta(1-p)[u_2(c_2) + \beta(1-p)u_3(c_3)], \tag{10}$$

where $c_1 := \bar{Y} - B + b_1 - Fb_1^+$, $c_2 := \bar{Y} - b_1 + b_2 - \min\{\rho R + p, R\}b_2^+$, $c_3 := \bar{Y} - b_2$. Given $b_2 = b_2^\eta$, lender profit function simplifies to

$$\Pi(F, R, L) := (F-p)b_1^+(F, R, L) + (1-p)\begin{cases} \rho R b_2^+(F, R, L) \text{ if } R > p/(1-\rho) \\ (R-p)b_2^+(F, R, L) \text{ otherwise.} \end{cases} \tag{11}$$

The consumer problem in this case is a standard convex programming problem that involves a strictly concave objective and continuously differentiable objective function defined on a convex and compact budget set. Importantly, as we show in the Appendix, a strictly monotonic transformation of the step-up rate and contract space leads to global differentiability in the redefined space. In terms of the redefined step up rate $\hat{R}$, given by equation (33) in the Appendix, consumer's budget constraint is $c_2 := \bar{Y} - b_1 + b_2 - \hat{R}b_2^+$, and the profit function of initial lender is $\Pi(F, \hat{R}, L) = (F-p)b_1^+ + (1-p)(\hat{R}-p)b_2^+$. The consumer refinances whenever $\hat{R} > p/(1-\rho)$.[20]

Proposition 1 characterizes the contract that solves the lender problem for $\eta = 1$. If local monotonicity applies, this contract is $F = p = R$, and it features a slack credit limit. This is a global maximum among contract that satisfy local monotonicity. Concluding, there is no promotional pricing in this case.

[20]Intuitively, this transformation exploits the property that the kink in the profit function and the consumer budget constraint is identical and thus can be "straightened" by a monotonic function with an appropriately defined offsetting kink.

Proposition 1 *Let (F, R, L) be an equilibrium contract in an economy with geometric consumers $(\eta = 1)$ and positive credit. Then, the allocation satisfies*

$$u'(c_1) = \beta u'(c_2), \tag{12}$$

and if, additionally, local monotonicity holds at (F, R, L), then i) $F = p = R$, ii) L is nonbinding, and iii) the consumer does not refinance, implying zero balance transfers.

To understand the intuition behind this result, consider first the optimality condition in (12). This condition naturally arises in a relaxed environment in which lenders can choose borrower's consumption in the first period and in the second period's high income state, yet must respect their zero profit condition and the fact that they cannot choose consumption in the low sate (default state) – in which case $c_2(s = 0) = \underline{Y}$. It is clear that lenders should choose consumption to equalize the marginal rate of substitution (MRS hereafter) between c_1 and $c_2 := c_2(s = 1)$ to the marginal rate of transformation (MRT hereafter) of funds across the periods. The fact that the consumer defaults with probability $1-p$ in the low income state trivially implies $MRT = -(1-p)$ (cost of funds is zero). Since consumer's consumption in the low income state is independent of her first period consumption and debt, the marginal rate of substitution between c_1 and c_2 is independent of consumption in the low state and hence $MRS = -(1 - p)\beta\frac{u'(c_2)}{u'(c_1)}$. Combining the two condition gives (12).

In our model lenders cannot choose consumption and must respect appropriate implementability constraints. The key implementability constraint is the borrower's Euler equation in the first period,

$$(1 - F)u'(c_1) = \beta(1 - p)u'(c_2). \tag{13}$$

Unlike (12), the consumer's Euler condition discounts the future at the probability of repayment $1-p$, which encourages borrowing in the first period. Consumers also take into account the distortionary effect of the first period rate F, which, in contrast, discourages borrowing. The first effect is a distortion from the lenders' perspective and second best contract that this setup delivers. This is because the cost of defaulting has to be paid by the consumer. Accordingly, lender would like to eliminate this distortion, and to do so, they price default risk to introductory rate by setting $F = p$, which boils down to choosing a flat rate schedule $F = p = R$ by the zero profit condition.

As an alternative, the contract could feature a binding credit limit L to implement (12). However, as we show, this cannot be optimal. Note that a binding borrowing constraint implies that raising F is nondistortionary up to $F = p$. Since R is distortionary, and the zero profit condition implies a negative trade-off between the two under LM, raising F up to $F = p$ is optimal. However, $F = p$ is the point that relaxes the borrowing constraint (lender profit in first period is zero regardless of L when $F = p = R$).

2.2.2 Hyperbolic consumers ($\eta < 1$)

We next consider hyperbolic discounting ($\eta < 1$) to show that a flat rate schedule $F = p = R$ no longer optimal; instead, the consumer receives a promotional offer with $F < p$. We also show that a sufficiently low value of the hyperbolic discount factor leads to zero APR "teaser" rates ($F = 0$) and refinancing in the second period. The result is summarized in Proposition 2.

Proposition 2 *Let (F, R, L) be the equilibrium contract in an economy with hyperbolic consumers ($\eta < 1$) and positive credit. Then, local monotonicity holds in the ex ante problem at $F = p = R$ (Lslack), we have $F < p < R$. Moreover, borrower's expectations of repricing are ex ante rational for (F, R, L), the allocation satisfies*

$$u'(c_1) = \beta\eta u'(c_2)\frac{b_2}{b_2^\eta}, \tag{14}$$

and, if $\frac{b_2}{b_2^\eta} \leq (1 - p)$, $F = 0$ and the consumer refinances for a sufficiently low value of ρ.

The reason why promotions are optimal when consumers are hyperbolic is because ex post borrowing b_2^η exceeds ex ante borrowing b_2 that consumers expect as of the first period (for any fixed first period borrowing b_1). This is clear from equation (8). Initial lenders exploit this systematic error to make contracts more attractive.[21] Formally, *lender optimality condition* is given by (14) and it additionally involves a wedge $\frac{b_2}{b_2^\eta}$, which, for any $\eta < 1$ is positive and can be made arbitrarily large by manipulating η and β. The intuition behind this wedge is that lenders receive an additional payoff from the error that consumers make. As before, implementation requires $(1 - F)u'(c_1) = \eta\beta(1 - p)u'(c_2)$, which gives $F < p$, and under certain conditions even $F = 0$. Since $F = 0$ and low ρ requires a high R to ensure zero profits, the consumer refinances in the second period.

2.2.3 Propagation of promo offers through chaining

Our next result shows that there is more to promotional offers than just the hyperbolic discount factor. It turns out that chaining of offers by itself triggers incentives to lower promotional rate F

[21] Consider an extreme example of a borrower who incorrectly assumes that she will be saving in the future while in actually she will borrow. The initial lender can exploit this fact by charging a sufficiently high step-up rate $R > p$ and a low promo rate F. Since the borrower believes she will never pay R, such will be considered attractive ex ante. The lender may nonetheless break even because the borrower will actually pay interest R ex post.

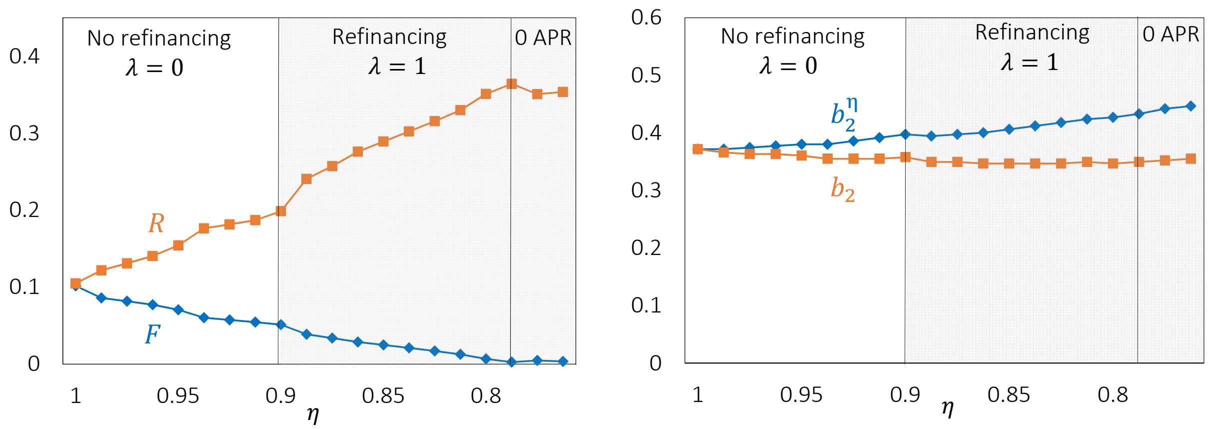

Figure 6: A numerical example: Equilibrium contract as a function of η ($\beta = 1$).

Notes: The figure illustrates equilibrium contract for a range of values of hyperbolic discount factor η, assuming $Y_l = 1/2, Y_h = 1, B = 1$, $\rho = .5$, $p = .1$, $\beta = 1$ and $u(c) = \log(c)$. F is restricted to be non-negative. The shaded area indicates when refinancing occurs on the equilibrium path. The right panel shows the wedge between ex ante and ex post borrowing that creates incentives to set promotional terms.

further. This mechanism arises even under geometric discounting, but since there is no refinancing, it has no bite.

To that end, consider a problem of a borrower who starts with an incumbent lender as of period one and assume this lender charges a step-up rate $R_{-1} > 0$ that is sufficiently high for the borrower to seek refinancing in the first period (as in the baseline case). Formally, let (5) be given:

$$c_1(B; b_1) = \overline{Y} - B(R_{-1}) - b_1 - \rho R_{-1} b_1^+ - F b_1^+, \tag{15}$$

where $B(R_{-1})$ is some function of $R_{-1} > 0$ such that, for equilibrium contract in economy with R_{-1}, income net of debt, $Y - B$, is exactly the same. This will allow us to compare the two economies. We denote the equilibrium contract in this parametric class of economies by $F(R_{-1}), R(R_{-1}), L(R_{-1})$. Note that $R_{-1} = 0$ trivially corresponds to our baseline economy.

The next proposition shows that in equilibrium initial lenders internalize the presence of R_{-1} and lower F further to "accommodate" the incumbent. The intuition is similar to that behind Propositions 1 and 2. Since the borrower's Euler equation is now distorted by $\rho R_{-1} + F$, rather than F alone, implementing the relaxed optimum is consistent with using $\rho R_{-1} + F$ to offset default-induced discounting, which is consistent with lower F.

Proposition 3 *Let $B(R_{-1}) := B - \rho R_{-1} b_1^+(0)$ be compensated debt, where $b_1^+(0)$ pertains to equilibrium borrowing in $R_{-1} = 0$ economy assuming initial contract $(F(0), R(0), L(0))$. Then, if borrowers* *expectations regarding ex post repricing are rational, i.e., then $F(R_{-1}) < F(0)$ as long as $F(0) \geq 0$ and $B(R_{-1}) > 0$.*

We finish by providing a numerical example illustrating the key results. We use the following functional forms and parameter values: $u = \log(c)$, $R_{-1} = 0$, $\underline{Y} = 1/2, \overline{Y} = 1, B = 1$, $\rho = .5$, and $p = .1$.

Figure 6 presents contracts and policy functions for different levels of η. As we can see, consistent with Proposition 1, $\eta = 1$ leads to $F = p = R$, and consistent with Proposition 2, lower and lower values of η are associated with lower and lower promotional rate F. Above $\eta = .9$ the step-up rate R becomes sufficiently high to trigger refinancing. As η falls below about .8, the consumer receives a zero APR offer with $F = 0$ (a corner solution).[22] Incidentally, this happens close to the estimated value of the hyperbolic discount factor by Ausubel and Shui (2005).[23] The right panel shows the wedge between the actual borrowing and ex ante (assumed) borrowing, i.e., b_2^η and b_2, which promotional offers exploit and which drives all these results. As we can see, it is not very large.

2.2.4 Macroeconomic implications of promo pricing: A discussion

The key lesson from the above characterization is that promotional activity shortens the effective maturity of credit card debt, as without refinancing rates simply go up. Therefore, it creates a macroeconomic vulnerability to disruptions of continual flow of credit. In particular, if the flow of credit is disrupted, and refinance opportunities do not arrive as expected, consumers have to pay R for the entire period as opposed to a fraction ρ of the period, where ρR and not R has set to cover the cost of past promotions. Our quantitative section will explore this mechanism in light of 2008-2014 deleveraging on credit cards.

3 Quantitative analysis

We turn to the quantitative analysis. We begin by laying out the generalized recursive setup that we take to the data. We then discuss parameterization and calibration, and turn to quantitative results.

[22] The corner solution follows from restricted domain to $F \geq 0$ – which we assume also our quantitative analysis. If there is no such bound, lenders will choose negative F for low values of η.

[23] Ausubel and Shui (2005) look at quarterly frequency. This implies that their estimated value $\eta = .81$ ($\beta \approx 1$) is rather conservative from the perspective of a model assuming a longer periods.

3.1 Generalized recursive setup

The key ingredients of the generalized setup are the same three-period model, but we additionally allow for endogenous default decision, multiple income states, random refinancing delays, and $T > 3$ periods long life-cycle. Restrictions on contracts are analogous. Timing within each period is also the same.

3.1.1 Consumers

Upon entering each period, consumer's initial state comprises her age t, open credit line $\mathcal{C}$, debt B, and the realization of Markov state $s = 0, 1, 2, .., n$ that determines her income and access to credit. At the onset of the period, after observing state s, the consumer receives the market offer and the repriced offer from the incumbent. As before, refinancing decision is denoted by $\lambda_t^\eta(\mathcal{C}, B, s)$ and occurs without recall.

Formally, let $M_t^\eta(\mathcal{C}, B, s)$ be the equilibrium *market offer* in state $(\mathcal{C}, B, s; t)$ and let $I_t^\eta(\mathcal{C}, B, s; \lambda)$ be the equilibrium repriced offer received from the incumbent depending on λ, since the incumbent reprices after the market closes and the consumer makes her switching decision (without recall). We define these equilibrium objects in the next section. Depending on the value of s, the consumer gains access to the market offer. In such a case, she chooses $\lambda^\eta(\mathcal{C}, B, s)$ to maximize

$$V_t^\eta(\mathcal{C}, B, s) = \max_{\lambda=0,1} U_t^\eta(M_t^\eta(\mathcal{C}, B, s), I_t^\eta(\mathcal{C}, B, s; \lambda), B, s; \lambda), \tag{16}$$

and otherwise $\lambda = 0$, where

$$V_t^\eta(\mathcal{C}, B, s) = U_t^\eta(M_t^\eta(\mathcal{C}, B, s), I_t^\eta(\mathcal{C}, B, s; \lambda), B, s; 0), \tag{17}$$

and $U_t^\eta(M_t^\eta(\mathcal{C}, B, s), I_t^\eta(\mathcal{C}, B, s; \lambda), B, s; \lambda)$ describes the indirect utility function conditional on λ.

Consider now the consumer problem after the lending market has closed (without recall). Let $\mathcal{C}_I^\eta := I_t^\eta(\mathcal{C}, B, s; \lambda) = (F_I^\eta, R_I^\eta, L_I^\eta)$ be the repriced contract extended by the incumbent lender after observing λ and let $\mathcal{C}_M^\eta := M_t^\eta(\mathcal{C}, B, s) = (F_M^\eta, R_M^\eta, L_M^\eta)$ be the market offer. The active offer is determined by the value of λ. The consumer chooses whether to default on accumulated debt during the period – as summarized by $\delta_t^\eta(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s; \lambda)$ – to maximize

$$U_t^\eta(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s; \lambda) = \max_{\delta=0,1} U_t^\eta(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s; \lambda, \delta). \tag{18}$$

The consumer chooses borrowing and consumption for the period – summarized by

$b_t^\eta(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s; \lambda, \delta)$ and $c_t^\eta(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s; \lambda, \delta)$ – to maximize

$$U_t^\eta(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s; \lambda, \delta) = \max_{(c,b)\in\Gamma} \{u(c) - \chi(s)\delta + \eta\beta\mathbb{E}_s[\delta V_{t+1}^1(\mathcal{C}_{-1}, 0, s') + (1-\delta)V_{t+1}^1(\lambda\mathcal{C}_M^1 + (1-\lambda)\mathcal{C}_I^1, b, s')]\}, \tag{19}$$

In the above problem, u is the utility function, $\chi(s)$ is state-dependent one time utility cost of defaulting on debt, and $\mathcal{C}_{-1} = (r_{-1}, 0, 0)$ corresponds to a seed contract that a consumer with no debt always starts with (where we set $r_{-1} = r$). Γ is the budget constraint defined by

$$c \leq Y_t(s) - B + b - (1-\delta)\left[\lambda F_M^\eta + (1-\lambda)(\rho F_I^\eta + F_M^\eta)\right] b^+ \tag{20}$$

$$b \leq (1-\lambda)\min\{L_M^\eta, L_I^\eta\} + \lambda L_I^\eta, \tag{21}$$

where, as before, b^+ is shorthand for $\max\{0, b^+\}$, $Y_t(s)$ is income, B is debt brought into the period and b is current borrowing.

As is clear from the budget constraint, if the consumer does not default ($\delta = 0$), consumption equals age-dependent income $Y_t(s)$ net of borrowing $B - b$ and interest payments. Interest payments are accrued in proportion to current debt b^+ and depend on the refinancing decision λ. In particular, if the consumer refinances, she pays $\rho F_I^\eta b^+$ to the incumbent lender on a fraction ρ of the period. $F_M^\eta b^+$ goes to market lender.

If the consumer defaults, and $\delta = 1$, debt is wiped out at the expense of an exogenous utility punishment $\chi(s)$. The punishment for defaulting χ depends on the state, and hence consumer income. This feature allows our model to quantitatively match the relatively high gross debt levels and default rates on credit card debt. Standard models are known to struggle in this respect. Drozd and Serrano-Padial (2017) provide micro-foundations for this formulation of punishment for defaulting.

3.1.2 Lenders

It is convenient to think about two types of lenders in our model: those who already an open credit line with a consumer and those who compete in the market to extend new offers. We refer to the former as *incumbent lenders* and to the latter as *market lenders*. In terms of notation, we drop dependence on policy whenever optimal policy is assumed. For example, we write $\delta_t^\eta(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s)$, to imply $\delta_t^\eta(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s; \lambda(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s))$. Similarly, we write $U_t^\eta(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s; \lambda(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s))$ to imply $U_t^\eta(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s)$.

Let $\Pi_t(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s)$ be the profit function of a market lender who extends offer $\mathcal{C}_M^\eta$ and let $\mathcal{C}_I^\eta$ be the equilibrium repriced offer extended by the incumbent lender. We define profit functions at the end of this section. The equilibrium *market offer* is

$$M_t^\eta(\mathcal{C}, B, s) = \underset{\mathcal{C}_M^\eta}{\operatorname{argmax}}\, U_t^\eta(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s) \tag{22}$$

subject to $\Pi_t^M(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s) = 0$.

Incumbent lenders reprice *after* the market offer is rejected or accepted by the consumer without recall. In case the consumer reprices, the incumbent lender still maintains the relationship with the consumer for ρ fraction of the period – during which interest rate is collected. The incumbent lender sets new terms under the legal restriction that: 1) repriced interest rate does not not exceed R, and 2) credit limit is sufficient to accommodate debt B brought into the period. In other words, consumers can opt out of any term changes and request $L = B$ and rate R, which is their outside option. Let C_M^η be the equilibrium market offer, let $\mathcal{C}_I^\eta = (F_I^\eta, R_I^\eta, L_I^\eta)$ be repriced offer by the incumbent and let $\mathcal{C} = (F, R, L)$ be pre-existing terms. The equilibrium *repriced offer* then is

$$I_t^\eta(\mathcal{C}, B, s; \lambda) = \underset{\mathcal{C}_I^\eta}{\operatorname{argmax}}\, \Pi_t^I(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s; \lambda) \tag{23}$$

subject to $R_I^\eta \leq R$, $F_I^\eta \leq R$, $L_I^\eta \geq B$, and $U_t^\eta(\mathcal{C}_M^\eta, \mathcal{C}_I^\eta, B, s; \lambda) \geq U_t^\eta(\mathcal{C}_M^\eta, \underline{\mathcal{C}}_I^\eta, B, s; \lambda)$, where $\underline{\mathcal{C}}_I^\eta = (R, R, B)$ is the consumer's opt-out contract.[24]

3.1.3 Lender profits

Before we formally define profit functions, note that consumers always defaults on at most B, regardless of what her credit limit on $\mathcal{C}$ was. As in the three-period model, the lender perfectly anticipates default after seeing consumer state and at repricing stage cuts the credit line as much as it is legally possible, which is B. The immediate corollary is that slack credit limits present no value to borrowers ex ante and can be assumed away without loss. This property simplifies the computation of equilibrium. We summarize these basic properties in the lemma below.

Lemma 4 *If the consumer defaults on incumbent's contract, $I_t^\eta(\mathcal{C}, B, s; \lambda) = (F_I^\eta, R_I^\eta, L_I^\eta)$ involves $L_I^\eta = B$. Without loss equilibrium market contract $M_t^\eta(\mathcal{C}, B, s) = (F_M^\eta, R_M^\eta, L_M^\eta)$ implies a tight borrowing constraint, i.e., $L_M = b_t^{\eta+}(M_t^\eta(\mathcal{C}, B, s), \mathcal{C}, B, s)$.*

The profit function $\Pi_t^I(\mathcal{C}_M^\eta, \mathcal{C}', B, s; \lambda)$ of the *incumbent lender* who repriced to $\mathcal{C}' = (F', R', L')$ is

[24] In the baseline setup, we assume that borrowers who discount the future hyperbolically expect that their future geometric self will be receiving an offer consistent with her preferences. Recall that this was our baseline case which from this point on we assume. Note, however, that our model can be converted to an alternative setup in which expectations are rational by taking out superscript η from M^η and I^η. We will not consider the alternative regime because we found that it makes little difference for quantitative results.

defined as follows: 1) if the consumer chooses not to refinance, i.e., $\lambda = 0$, and defaults in the current period, i.e., $\delta_t^\eta(\mathcal{C}_M^\eta, \mathcal{C}', B, s; 0) = 1$, lender profit is $\Pi_t^I(\mathcal{C}_M^\eta, \mathcal{C}', B, s; 0) = -B(1+r)$; 2) if, on the other hand, the consumer chooses not to refinance, i.e., $\lambda = 0$, and repays, i.e., $\delta_t^\eta(\mathcal{C}_M^\eta, \mathcal{C}', B, s; 0) = 0$, it is[25]

$$\Pi_t^I(\mathcal{C}_M^\eta, \mathcal{C}', B, s; 0) = (F' - (1-\rho)r) b_t^{\eta+}(\mathcal{C}_M^\eta, \mathcal{C}', B, s; 0) + \mathbb{E}_s[\bar{\Pi}_{t+1}^I(\mathcal{C}', b_t^{\eta+}(\mathcal{C}_M^\eta, \mathcal{C}', B, s), s')]/(1+r),$$

where the continuation value is the profit of an *incumbent lender* as of the onset of the next period; that is, $\bar{\Pi}_t^I(\mathcal{C}', B, s) := \Pi_t^I(M_t^\eta(\mathcal{C}', B, s), I_t^\eta(\mathcal{C}', B, s), B, s)$. Finally, 3) if the consumer chooses to refinance, i.e., $\lambda = 1$, the incumbent lender collects interest for fraction ρ of the period, implying $\Pi_t^I(\mathcal{C}_M^\eta, \mathcal{C}', B, s) = \rho(F' - r) b_t^{\eta+}(\mathcal{C}_M^\eta, \mathcal{C}', B, s; 1)$. (If the borrower refinances and defaults, the profit of the incumbent is zero.)

The profit function $\Pi_t^M(\mathcal{C}', \mathcal{C}_I^\eta, B, s)$ of a *market lender* whose contract $\mathcal{C}' = (F', R', L')$ has been accepted by the borrower, i.e., $\lambda(\mathcal{C}', \mathcal{C}_I^\eta, B, s) = 1$, is defined analogously to the profit function of the incumbent lender who reprices and whose offer has been accepted; that is, $\Pi_t^M(\mathcal{C}', \mathcal{C}_I^\eta, B, s) = \Pi_t^I(\mathcal{C}_I^\eta, \mathcal{C}', B, s; 1)$. If the consumer does not accept the market offer $\mathcal{C}'$, i.e., $\lambda(\mathcal{C}', \mathcal{C}_I^\eta, B, s) = 1$, profit is zero.

3.1.4 Recursive equilibrium

Recursive equilibrium comprises consumer's policy functions c_t^η, b_t^η, δ_t^η, lender pricing policies M_t^η, I_t^η, and consumer and lender value functions V_t^η, U_t^η, Π_t^I, Π_t^M consistent with (16), (18), (19), (22), (23), and the definitions of Π_t^I and Π_t^M in Section 3.1.3.

3.2 Parameterization

This section describes the choice of functional forms.

We assume time-invariant log utility, $u(c) = \log(c)$. The cost of defaulting χ depends on agent's income and we parameterize as follows: $\chi(y) = \chi_1 \max(y - \chi_0, 0)$, where y is agent's current income flow. In the interest of reducing the number of parameters, we assume χ_1 is arbitrarily large so that raising it would not affect our results and use χ_0 in calibration. This assumption implies that below a certain income level households always default and otherwise they repay. χ_0 is jointly calibrated

[25]Note that the cost of funds for market lender is $(1-\rho)r$, since the incumbent lends to her during the initial ρ fraction of the period. Had this cost not been here our model would involve a cost of refinancing, since the cost of funds would be higher in case of refinancing.

with other parameters to key moments characterizing the credit card market in 2007, as described in Section 3.2.

The value for the hyperbolic discount η is taken from Ausubel and Shui (2005), who report $\eta = 0.81$ in a quarterly model. We are conservative in assuming that there is no further hyperbolic discount that applies beyond one quarter. The geometric discount factor β is jointly calibrated with other parameters to match the key moments characterizing the credit card market in 2007, as we describe next.

The key input to the consumer problem is the income process. Our framework takes into account that income process may be different during recessions and expansions. Specially, income during working age of an agent in state of the economy ω is governed by $y_t(\omega) = e_t k_t z_t(\omega)$, where y_t is agent's income at age t, e_t is deterministic age-dependent income profile, k_t is a 3 state discrete i.i.d. process and $z_t(\omega)$ is a 6x6 discrete state Markov process that depends on a binary Markov process ω such that *recession state* ($\omega = 0$) occurs with probability p. We assume that individuals on average retire with a replacement rate of 75 percent, which is within range of estimates by Munnell and Soto (2005).[26]

Individuals in the model are assumed to start their life at the age of 24 years, retire at the age of 65 year, and die at the age of 80 years. Model period length l is one of the parameters that we calibrate. Depending on its value, our model assumes deterministic life-cycle of a total of (80-23)/l periods and (65-23)/l retirement periods.

In terms of lending technology, the parameter ρ and the lender cost of funds r are jointly calibrated with other parameters to match the key moments characterizing the credit card market in 2007, as described in Section 3.2. The period length is part of calibration so that the model delivers offer arrival delay ρ of a particular length. Consistent with our data, the median duration of promo spell on credit cards is 12 months, and hence we impose a structural restriction that $\rho l = 1$ (year). This implies that there is a single degree of freedom in choosing the two parameters. ρ crucially affects the size of the step-up rate on credit cards, which we use to calibrate it. Note that a very low ρ implies that the step up rate applies to a very short period of time. If promotional offers are used, the step up rate must be large in such a case for lenders to break even.

[26]Retirement income involves no shocks and it is a combination of income prior to retirement (35 percent weight) and a constant equal to 40 percent of the average income in the economy (or median, since these are close in the model).

3.3 Calibration

This section describes how we choose parameter values.

3.3.1 Income process

We start from the continuous space regime dependent processes estimated by Guvenen et al. (2014) (model 4, table 1). Since the period length in our model is part of calibration, we switch the frequency of the autoregressive processes reported by Guvenen et al. (2014) to l months, where l is the assumed period length in our model. We generate five processes that span the range between $l = 12$ and $l = 24$ and discretize each of them by following the same procedure. An analogous procedure is applied to the i.i.d. component of the process.[27] We switch to the nearest process as we vary l in moment matching calibration procedure.

3.3.2 Remaining parameters

We choose the values of $\beta, \chi_0, l, \rho, r$ jointly to match the following moments characterizing the U.S. credit card market in 2007 (early 2008 if micro data is used): 1) credit card debt per adult with a card (78%) relative to the median income of 22 percent,[28] 2) the net chargeoff rate on credit card accounts of 4 percent (top 100 banks), 3) the average duration of a promo spell of 12 months, 4) the average step-up rate on credit card plans that involve a promotional rate or in the past of 17.5 percent, 5) the average debt-weighted rate paid on credit card debt of 11.9 percent (APR) in our micro data. Table 2 lists targeted values and the obtained values of the parameters. These targets come from our aggregate data source except for the last one, which we calculate using our micro data

[27]We discretize the AR1 processes as follows. We assume 6 states and set five grid points for the Markov process at the cutoffs corresponding to the following percentiles of the sample path of the joint AR process with regime switching: $\{10, 25, 50, 75, 90\}$. We chose these cutoffs to most accurately describe shocks affecting lower income consumers. For the i.i.d. process k_t, we assume a single negative and a single positive value. Both shocks require a deviation of at least 15 percent from the mean value. The size of the shock is the average value of such conditional deviation. Finally, the deterministic age profile of income is obtained using the standard regression model that includes age, sex, and a polynomial on education dummies. We use PSID data since we do not have access to the data underlying the estimated process by Guvenen et al. (2014). The data and regression specification follows Karahan and Ozkan (2013). To focus on the lower income tiers, we remove top 5% income earners from this regression, as it has been shown by Guvenen et al. (2016) that top income earner life-cycle profiles are much steeper and joint estimate is poor match for a median. The parameterization of the process is too large to report here but it can be found in supporting files and the Online Appendix.

[28]We consider median rather then average income to be a better target. In the model median income and average income are approximately equal. As is well known, this is not the case in the data due to a disproportionate share of the top income bracket (e.g. the top one percent). We use Social Security Administration median net compensation, `https://www.ssa.gov/oact/cola/central.html`. The fraction of adults with a card is calculted using credit card adoption rate reported by The 2008 Survey of Consumer Payment Choice (page 16), Foster et al. (2008).

for first quarter of 2008 (for aggregate data we use the average value for 2007). (For further details on data sources, see Appendix A.)

Table 2: Data targets and calibrated values of jointly selected parameters.

	Data	Model
A. Targeted moments		
1. Credit card debt of card holder to median personal income [%]	22	21
2. Net charge-off rate [%]	4.0	4.1
3. Average duration of promo offers [months]	12	12
4. Average step up rate on promo accounts [%]	17.3	18.4
5. Average rate on credit card debt [%]	11.9	11.9
B. Jointly calibrated parameters		
Discount factor β		0.926
Cost of defaulting χ_0		0.867
Period length l [months]		20
Refinance delay ρ		0.4
Lender cost of funds r		0.07
C. Preset parameters		
Hyperbolic discount factor β		0.81
Probability of receiving refinance offer		1.00
Income process (see Online Appendix and supp. files)		

Notes: Panel A presents key moments we used to calibrate the parameters listed in panel B, while presenting the parameters in panel C. The calibration is for year 2007 which in model follows a 3 period economic expansion. Revolving credit (G.19) published by the of the Federal Reserve Board of Governors (FRB) is our baseline measure of credit card debt in aggregate (933,140 millions of 2007 dollars). We divide this number by the number of adults 21 or older (216 million) published by United Nations, Department of Economic and Social Affairs, World Population Prospects: The 2017 Revision. We assume 78 percent of persons have a credit card, consistent with the 2008 Survey of Consumer Payment Choice published by the Federal Reserve Bank of Boston. Income is median net income published Social Security Administration. To calculate the first moment, we divide the obtained debt per person with at least one card by the median income of about $25,737 in 2007 dollars. The second moment is the net charge-off rate published by FRB for 100 largest banks (SA). It is calculated as an average for 2007. The remaining moments are for 2008q2 account-level micro data described in Section 1.

We calibrate the model by minimizing the squared distance between model implied moments and the data. In the model we assume that the economy is in expansion for 5 years before 2007 ($5/l$ model periods). The calibration of individual parameters cannot be separately mapped onto individual moments. Sensitivity analysis shows that β and χ_0 are crucial to match targets 1 and 2; period length l and the delay parameter ρ is crucial to match targets 3 and 4; r is crucial to match 5, though it affects 4. In the model we assume that the economy is in expansion for three consecutive model periods.[29]

[29]In the prior periods we simulate the model by assuming business cycle regime switches are uncorrelated across agents.

3.4 Model validation

Since our model is not saturated, some moments that are relevant for our analysis remain endogenous. We report them in Table 3. It is important that our model comes close to matching those crucial moments.

Table 3: Data targets and calibrated values of jointly selected parameters.

Statistic (in percent % unless otherwise noted)	Data	Model
Promo debt as a fraction of total debt	35	33
Annual balance transfers as fraction of debt	39	44
Average interest rate on promo debt (+3 in data)	7	6
Median interest rate on promo debt (+3 in data)	6	6
Share of revolvers among card users	59	60

Notes: Data as described in Section 1. Share or revolvers among card users is approximated using the share of card users of 78 percent reported in Foster et al. (2008) and 46 percent share of households reporting credit card debt according to Foster et al. (2012). The reported value .59 satisfies the formula: $(1 - .22)0 + .78 \times .59 = 0.46$. Interest rates on promo balances in data include an imputed 3 percent balance transfer fee.

The most important one is the fraction of debt that has promo status. We have not targeted this moment and it is important that the mode is reasonable in this dimensions. The model has more dispersed promotional rates than we see in the data and hence using a definition of promo rate as accounts with $F < R$ leads to a much higher level of average promotional rate in the model than it is in the data.[30] We use a more restrictive definition of promotional debt that requires at least 50 percent discount relative to the step-up rate; i.e., $F < .5R$ is promo in the model. The model implies 33 percent share of promo debt according to this definition, versus 35 in the data. For comparison, if an analogous condition of 50 percent discount is imposed on the data, we obtain that the share of such promo debt to total debt falls to 24 percent. However, the distribution is much more tightly centered around the mean and median, while in the model discounts are dispersed. Concluding, the model is consistent with data as far as large discounts go, but it implies frequent small promotions that are far less prevalent in the data.

The second key moment is the average gap between the promotional rate and the step up rate, or equivalently the average promotional rate given we targeted the step up rate. In this respect the model does well. This is not surprising given our definition of promotional account.

The average promotional rate we find in the data is a little over 4 percent and the median is less

[30] The frequency of unconditional promo rate defined by $F < R$ is 58 percent in the model, and hence much higher than it is in the data.

than 4 percent. Imputing a typical balance transfer fee of 3 percent[31] over the average 12 months-long duration of promotional period we find in the data, gives an annual rate of about 7 percent over the mean and 6 percent over the median. The model implies a little over 6 percent over the mean value.

The third key moment is how much debt a statistical card holder has. This number has been matched in the calibration but it can be decomposed to an extensive margin, namely the fraction of revolvers, and the intensive margin, the amount of debt per revolver. As the table shows, the model matches estimated data value.

4 Analysis of credit card deleveraging

We now discuss our leading experiment of simulating the impact of a recession and the collapse of promotional rates.

4.1 Setup

The first period in our simulation corresponds to year 2007 and hence matches the pre-crisis statistics that we used in the calibration of model parameters. Since frequency in our model is non-standard, as it is part of calibration, the subsequent periods are 20 months apart: 2008/9, 2010/5, 2012/1, 2013/9, 2015/5, defining model period intervals in-between. To map the model to annual data, we take 20-months long model periods centered around these dates and construct annual series as follows: if a given year falls into one of these intervals, it takes the value from this interval, if it falls across model periods, it is a weighted average of the values from the two model periods depending on the fraction of the year that falls into each period.

In our simulation, we make sure we are consistent with the U.S. demographics. This is challenging in a deterministic life-cycle model. To that end, we start from random sample of millions of agents consistent with the age-structure consistent in 2007 (we use 2010 Census Bureau weights). We then assume that each year individuals dies with an age-dependent probability taken from the actuarial life tables published by U.S. Social Security Administration (year 2010). Deceased agents are replaced by newborn agents. Since this yields zero population growth, we add newly born agents to match 0.9 percent annual population growth.

[31] See page 187 of "The Consumer Credit Card Market," Consumer Financial Protection Bureau, 2017, available at https://files.consumerfinance.gov/f/documents/cfpb_consumer-credit-card-market-report_2017.pdf.

In our baseline scenario, we shut down the probability of receiving new offers to match closely declining share of promotional debt. We also assume that as of 2008 agent part of the shock that we find must remain at the end of the sample is a permanent regime shift. The permanent decline in the probability of receiving refinance offer is from 1 to .66. The unexpected decline the residual needed to match the evolution of the data series in between.

We refer to the first part of the shock as *transition* and the second part as *collapse of promo*, although both jointly model collapse of promo in the data. The motivation for this being the baseline case is that thus far we do not see a full rebound of promotional activity in the data. We believe the assuming that the shock that lasts almost a decade is in part expected is also more reasonable. The Online Appendix considers a fully unexpected shocks and the results are almost the same.

The model assumes that in the second period a recession starts and lasts for two model periods in total, hence longer than the NBER recession. A recession results in a switch to a recessionary income process across all agents in the economy. The extended recessionary interval helps the model to match the evolution of the chargeoff rate and is generally consistent with a far more persistent decline in consumption and income seen in the data. Such extended dates are also consistent with Guvenen et al. (2014), whose regime dependent income process we used in calibration. We report results from the model by decomposition the contribution of each channel.

4.2 Findings

We now discuss our findings.

4.2.1 Collapse of promo

Figure 7 shows how our assumed shock matches the key premise of our analysis: the decline in the share of promotional debt and the associated with it collapse of balance transfers. As we can see, the evolution of share of promo debt is closely matched and balance transfers are matched peak-to-trough, after which they recover in the model but not in the data for the sample of banks we focus on. Since balance transfer for all banks have recovered more, and hence our model understates the shock, we consider this discrepancy acceptable.

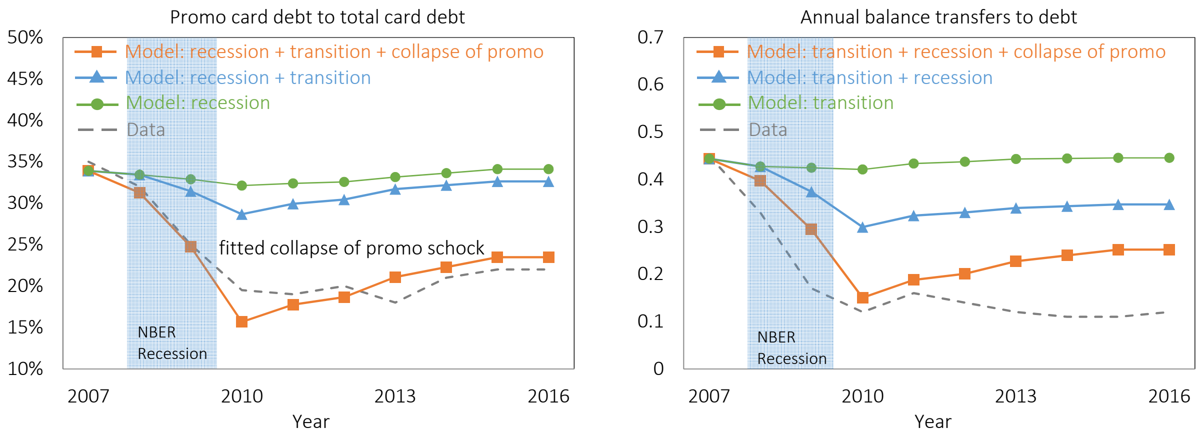

Figure 7: Collapse of promotional activity: model via-á-vis the U.S. data.

Notes: The figure illustrates the decline in the share of promotional credit card debt to total debt (left panel) and the collapse of balance transfers (promotional balance transfers) as a fraction of debt. Solid lines correspond to the model and the dotted line is the data. We consider three models that incrementally add shocks. The total contribution of the collapse of promo shock is the difference between green line with circles and the orange line with squares.

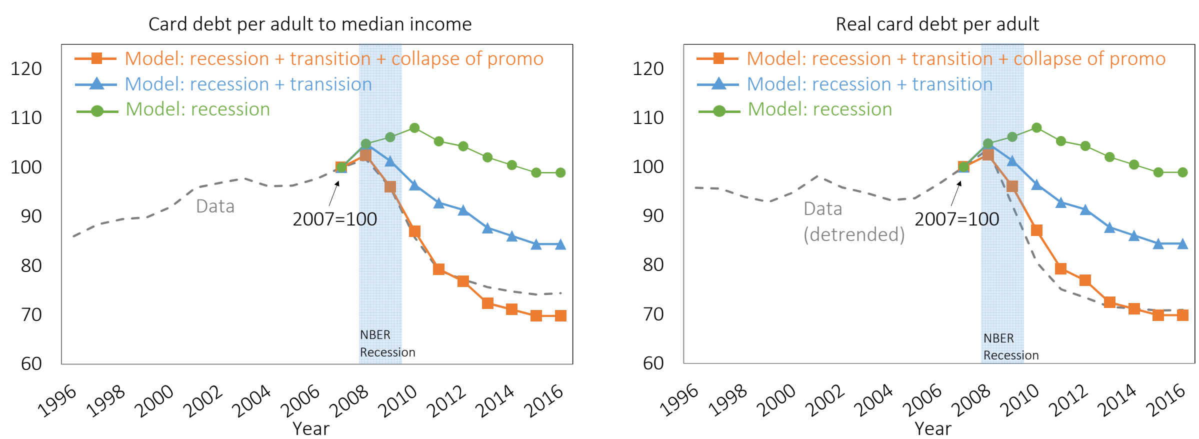

Figure 8: Deleveraging on credit cards: model via-á-vis the U.S. data.

The figure illustrates deleveraging on credit cards relative to median income and in absolute terms (in data real value detrended using the 1996-2006 linear trendline). Solid lines correspond to the model and the dotted line is the data. We consider three models that incrementally add shocks. The total contribution of the collapse of promo shock is the difference between green line with circles and the orange line with squares.

4.2.2 Deleveraging

Figure 8 shows deleveraging on credit cards. Here we use aggregate revolving credit series published by Federal Reserve Board of Governors (FRB), and hence relate our model to the entire market. The left-panel plots revolving debt relative to median income (SSA median net compensation). The right-panel focuses on detrended real revolving credit. As we can see, dividing by income makes little difference and it is a natural way of detrending the data. In both cases the model matches remarkably closely the data, slightly overshooting the depth of deleveraging. This highlight the potency of the channel we stress. Since our calibration slightly overstates the importance of promotional debt we consider this discrepancy acceptable. The important take away is that deleveraging is almost exclusively accounted for by the collapse of promo, which corresponds to the difference between the orange line with squares and the green line with circles. The recession itself leads to slightly more debt, as should be expected in this class of models (blue line).

4.2.3 Charge-off rate and interest rate on card debt

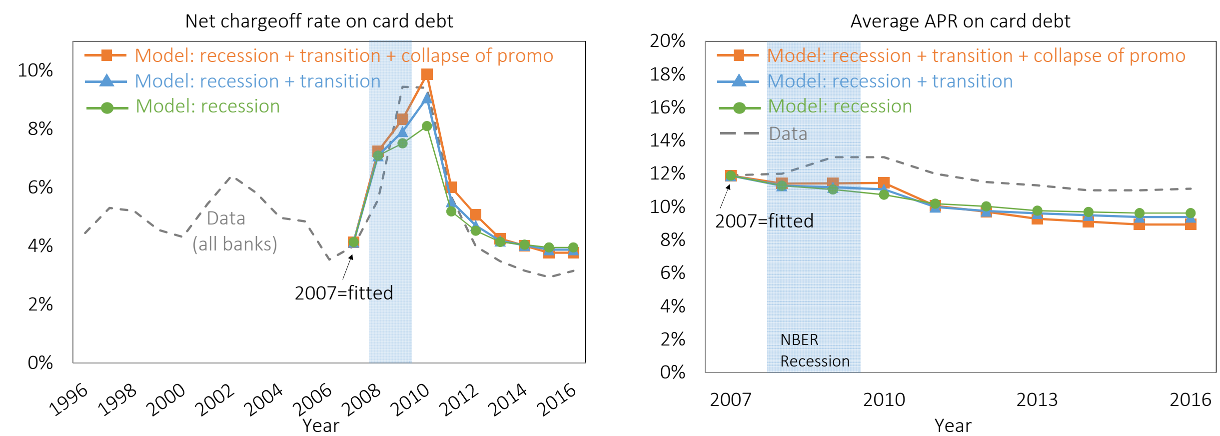

Figure 9: Charge-off rate and interest rate on card debt: model via-á-vis the U.S. data.

Note: The figure illustrates the net charge-off rate on card debt (fraction of debt defaulted on) and the average interest rate paid on credit card debt estimated using our account level dataset. Solid lines correspond to the model and the dotted line is the data. The charge-off rate is for all banks and comes from FRB. We consider three models that incrementally add shocks. The total contribution of the collapse of promo shock is the difference between green line with circles and the orange line with squares.

Figure 9 (left-panel) focuses on the net charge-off rate on credit card debt for all banks (also from FRB). The charge-off rate measures the fraction of debt that is charged off by banks due to persistent delinquency, net of the flow of recoveries. As we can, the model closely matches its evolution, but in this case it is the recession shock that plays the dominant role. The collapse promo adds at most 15 percent at the highest point.

In the model deleveraging is caused by borrowers who suddenly cannot refinance their debt using a promotional offer and repay or default in fear of having to pay the much higher step-up rate in the future period(s). Accordingly, an important check on the consistency of the model is whether it can match the relatively modest increase in the average interest rate paid on credit card debt seen in the data. The interest rate went up by about 100 basis point for a short period of time, and overall has been slightly declining, albeit much less than the prime rate. As Figure 9 (right-panel) shows, the model predicts an even smaller increase and some secular decline due to changing demographics. The key mechanism behind this result is the selection effect: Those who repay the debt or default are those who face the largest hike in interest rate.

4.3 Aggregate implications

An important question is whether the collapse of promotional activity had aggregate consequences and play a role in the Great Recessions. While our model requires some adjustments to take it to aggregate data – since we calibrated it to median income – we can shed light on this question. We will focus on consumption as a way to assess aggregate demand implications of our model.

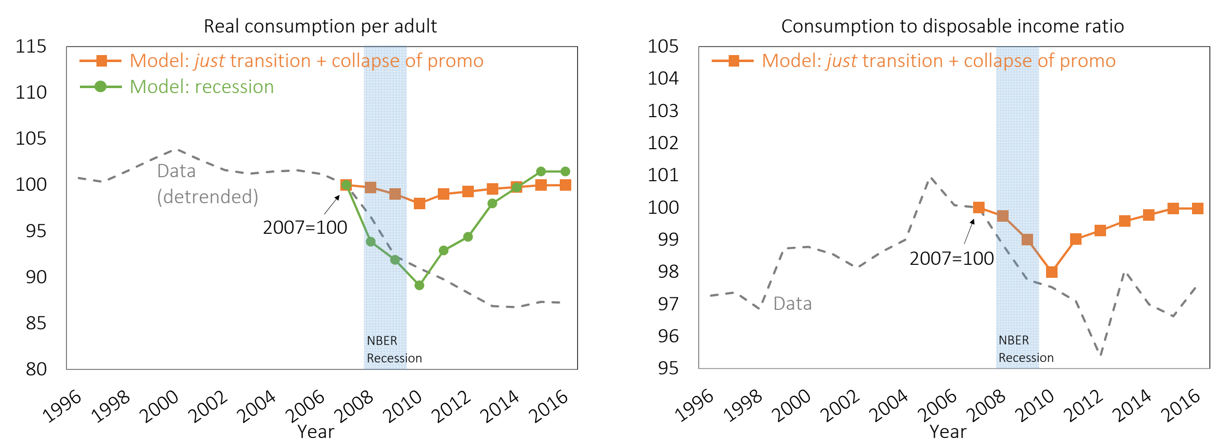

Figure 10: Charge-off rate and interest rate on card debt: model via-á-vis the U.S. data.

Note: The figure illustrates the net contribution of the collapse of promo shock to decline in consumption in the model (orange line with squares). Solid lines correspond to the model and the dotted line is the data. The left-panel looks at total consumption and the right panel looks at consumption income ration (aka average propensity to consume). The orange line here refers to net added contribution of having promo shock (with transition) and isolates out this shock from recession and demographic changes.

Figure 10 (left-panel) shows the effect of transition and collapse of promo when isolated from all other effects on real aggregate consumption in the model. As we can see, there is little effect: much of the decline in consumption is due to declining income and only 10 percent of the decline is due to the collapse of promo shock. Our model implies very little difference between median and average income, and hence 10 percent is an overstated contribution in light of the data. Roughly, the ratio of median income we use and disposable income in 2007 is 26/45 and similar in the subsequent years. Assuming income unaccounted for is not affected by the collapse of promo shock, collapse of promo shock would not account for more than 5.5 percent of the overall decline in consumption between 2008-2011.

However, if it is consumption that is causing recession and the fall in income, which, in turn, lowers consumption, as argued by Mian and Sufi (2014), looking at the aggregate consumption is not the right metric. As a imperfect measure, in such a case a better metric would be the ratio of consumption to disposable income, also known as the *average propensity to consume.* Figure 10 (right-panel) provides such a comparison. As we can see, the isolated effect of promo shock generates a decline that is roughly half of the one we see in the data between 2007 and the trough in 2012. As before, relating the model to aggregate data requires a correction because the model is calibrated to median income and median income is equal to average income in the model but not in the data. As a result, our model roughly leaves out about 40 percent of income concentrated among top earners. Using the same back of the envelope scaling by factor 26/45, which is the ratio of median income and disposable income in the data (in 2007), still implies that the decline in the model is sizable.

The Great Recession had been propagated by declining investment, public spening, and other factors, but if consumption indeed played a key important role, as Mian and Sufi (2014) find, the above calculation suggests that the collapse of promotional activity in the credit card market importantly contributed to deteriorating demand conditions in the economy after 2008.

We conclude by noting that the shock we document is quantitatively relevant for understanding the credit card market during and after the crisis and also the dynamics of aggregate consumption after 2008, especially in light of the particular demand channel stressed by Mian and Sufi (2014).

References

Agarwal, S., S. Chomsisengphet, C. Liu, and N. Souleles, "Do Consumers Choose the Right Credit Contracts?," *Review of Corporate Finance Studies*, 2015, *4*, 239–257.

Agarwal, Sumit, Souphala Chomsisengphet, Neale Mahoney, and Johannes Stroebel, "Regulating Consumer Financial Products: Evidence from Credit Cards," *Quarterly Journal of Economics*, 2015, *130* (1), 111–164.

Ausubel, Lawrence M. and Haiyan Shui, "Time Inconsistency in the Credit Card Market," *Working Paper, University of Maryland, Department of Economics*, 2005.

Drozd, Lukasz and Ricardo Serrano-Padial, "Modeling the Revolving Revolution: The Debt Collection Channel," March 2017, *107*, 897–930. American Economic Review.

Evans, David S. and Richard Schmalensee, *Paying with Plastic: The Digital Revolution in Buying and Borrowing*, The MIT Press, 2005.

Foster, Kevin, Erik Meijer, Scott Schuh, and Michael A. Zabek, "The 2008 Survey of Consumer Payment Choice," Working Paper 09-10, Federal Reserve Bank of Boston April 2008.

_ , _ , _ , and _ , "Changes in U.S. Family Finances from 2007 to 2010: Evidence from the Survey of Consumer Finances," Bulletin 2, Federal Reserve Board of Governors 2012.

Grodzicki, Daniel, "Competition and Customer Acquisition in the U.S. Credit Card Market," *mimeo Penn State University*, 2015.

Guvenen, Fatih, Fatih Karahan, Serdar Ozkan, and Jae Song, "What Do Data on Millions of U.S. Workers Reveal About Life-Cycle Earnings?," *mimeo University of Minnesota*, 2016.

_ , Serdar Ozkan, and Jae Song, "The Nature of Countercyclical Income Risk," *Journal of Political Economy*, 2014, *122* (3), 621–660.

Karahan, Fatih and Serdar Ozkan, "On the Persistence of Income Shocks over the Life Cycle: Evidence, Theory, and Implications," *Review of Economic Dynamics*, 2013, *16*, 452–476.

Laibson, David, Andrea Repetto, and Jeremy Tobacman, "Estimating Discount Functions with Consumption Choices over the Lifecycle," 2007.

Mian, Atif and Amir Sufi, "What explains the 2007-2009 drop in employment?," *Econometrica*, November 2014, *82* (6), 2197–2222.

Mian, Atif R and Amir Sufi, "Household Leverage and the Recession of 2007 to 2009," Working Paper 15896, National Bureau of Economic Research April 2010.

Munnell, Alicia H. and Mauricio Soto, "What Replacement Rates Do Households Actually Experience in Retirement?," *Center for Retirement Research Working Paper 2005-10*, 2005.

Appendix

A. Data sources and supplementary tables and figures

Definitions of variables in OCC/Y14M sample

Here we present the definitions of the variables we used in Section 1 in Table 1 and figures 2,3 and 5. This data comes from the OCC/Y14M sample.

Our main variable is debt defined at a month-account level. We calculate it as a maximum of zero and a difference between the cycle ending balance from the previous month and total payments made in a current month. Total debt in a quarter used in Section 1 is a sum of debt on all accounts in a given quarter. In the first quarter of 2008 the total debt encompasses only months of February and March because we define debt based on the balance lagged by one month.

Our next main variable is "promo debt": debt residing on a promotional account in a given month. Prior to June 2013 we identify an account as promotional when the reported variable "promotional balance" is positive. After June 2013 our sample contains a variable "promotional flag", which is used to identify the promotional accounts. We also identify promo debt nearing expiration as debt in the last month before an account stops to be promotional (i.e., the promotional balance turns from positive to zero).

Promo spell is a number of months for which an account has a positive promotional balance.

Because the APR for promotional balances is not reported prior to June 2013, for comparison reasons we calculate APR for all types of accounts in two ways. First, we calculate average APR (in Table 1 and figure 3) using the method used by the Federal Reserve in their Consumer Credit G.19 schedule. This method imputes APR from the aggregate numbers for dollars of finance charges and outstanding balances. More specifically, APR for given type of accounts (all, promo, non-promo) in a given quarter is a ratio of sum of finance charges to sum of outstanding balances for all accounts of a given type. For promo accounts we take into account that some balances are not on a promotion and we subtract an imputed finance charge on these balances from total finance charge and reported APR on non-promotional balances.

In Table 1 we also report median debt-weighted APR for account of all types. For non-promo accounts we can simply use the reported APR and then weigh it by debt. For promo accounts we have to impute APR on a monthly level from the reported finance charges and outstanding balances. Since the imputed APR for promotional accounts is relatively noisy due to reporting issues we report only a median after winsorizing data.

Finally, we compute the step-up rate as the APR 2 months after expiration of the promotional status of the accounts. We identify this APR with a 2-month lag to make sure that we are capturing the expiration of the promotional status correctly.

Other data sources

Figures 1,5, 7-10 and Tables 2-3 use the following publicly available data series: 1) total card debt corresponds to revolving consumer credit: Board of Governors of the Federal Reserve System (US), Total Revolving Credit Owned and Securitized, Outstanding [REVOLSL], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/REVOLSL, October, 2018. 2) Real card debt is revolving consume rcredit defalated by CPI (2007=1): U.S. Bureau of Labor Statistics, Consumer Price Index for All Urban Consumers: All Items [CPIAUCSL], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/CPIAUCSL, October, 2018. 3) Median income corressponds to SSA median net compensation retrived from Social Security Administration; https://www.ssa.gov/OACT/COLA/central.html, October, 2018. 4) Real consumption corresponds to real personal consumption expenditures per adult (21 years or older): U.S. Bureau of Economic Analysis, Real Personal Consumption Expenditures [PCECCA], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/PCECCA, October 25, 2018 and populatio data from United Nations, World Population Prospects 2017, retrieved from https://population.un.org/wpp/DataQuery/. 5) Consumption to disposable income uses ratio of nominal series: U.S. Bureau of Economic Analysis, Personal Consumption Expenditures [PCECA], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/PCECA, October, 2018 and U.S. Bureau of Economic Analysis, Disposable personal income [A067RC1A027NBEA], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/A067RC1A027NBEA, October, 2018. 6) Consumer credit less student loans: Board of Governors of the Federal Reserve System (US), Total Consumer Credit Owned and Securitized, Outstanding [TOTALSL], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/TOTALSL, October, 2018, and Quarterly Report on Household Debt and Credit, August 2018, Federal Reserve Bank of New York Research and Staticistics, Microeconomic Studies, retrived from https://www.newyorkfed.org/microeconomics/databank.html, October, 2018. 7) Net chargeoff rate: Board of Governors of the Federal Reserve System (US), Charge-Off Rate on Credit Card Loans, All Commercial Banks [CORCCACBS], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/CORCCACBS, October, 2018.

Linear probability model used in Section 1

Table 4 shows estimated a linear probability regression model (LPM) using credit bureau data. We estimated the model using data from 2002 to 2007, and hence prior to the Great Recession. We focused on cardholders with at least $3000 in total credit card balances and less than $100000.

Table 4: Linear model of the annual number of balance transfers.

Explanatory variable	Regression coefficient
Highest revolving balance	3.88e-6 (2.3e-7)
Bank card balances (total)	-9.6e-7 (9.0e-8)
Bank card utilization	-0.0245 (0.0005)
Bank card total monthly min payments to balances	-0.0057 (0.0021)
Severe bankcard delinquency on record	-0.0647 (0.0028)
Bankruptcy on record (indicator)	-0.0055 (0.0044)
Severe bank card delinquency in next 24 months (indicator)	-0.0216 (0.0031)
Bankruptcy in next 24 months (indicator)	-0.0222 (0.0063)
Recent bank card delinquency on record (last 24 months)	-0.0368 (0.0036)
Recent major credit card delinquency on record (last 24 months)	-0.0156 (0.0055)
Bankcard accounts ever discharged or settled (number)	0.0003 (0.0008)
Bank card debt recovery score	-0.0002 (0.0000)
Current Vantage v3 credit score	-0.0006 (0.0000)
Home equity lines (number, open)	0.0512 (0.0020)
Revolving accounts (number, open)	0.0307 (0.0003)
Auto loans (number, open)	0.0017 (0.0012)
Mortgage (indicator, open)	0.0113 (0.0019)
Mortgage balance (open, total)	1.23e-8 (6.4e-9)
Court judgments all (number on record)	-0.0957 (0.0800)
Foreclosures all (number on record)	-0.0734 (0.0504)
Collection inquiries all (number on record)	-0.0094 (0.0017)
Year FE x State FE	
constant	-5.1049 (0.7160)
N	313,038
R2	0.0668

B. Omitted proofs (extended version)

As a tie breaking rule, we will assume the incumbent lender will adjust credit terms at repricing stage if the consumer's utility can be strictly increased without lower the profits. Utility function u is assumed to be continuously differentiable, strictly increasing and strictly concave.

Proof of Lemma 1.

As discussed in text, the market lender will always relax the second period borrowing constraint, implying L_M never binds. An analogous argument applies to L_I: 1) If the borrower refinances, since the incumbent bears no default risk, more borrowing only raises her profits for any positive repriced second period interest rate F_I. Suppose then that the rate is negative, i.e., $F_I < 0$. Clearly, this cannot be optimal since the lender has a monopoly over the borrower and setting $F_I = 0$ for any $b_2 > 0$ would only increase her profit. If $b_2 < 0$ the constraint does not bind. 2) If the borrower, in turn, stays with the incumbent, the incumbent's continuation profit is the same as new lender's, i.e., $(F_I - p)b_2^+$. Hence, setting a binding credit limit would only make sense if $F_I < p$. Unless $R < p$, again, the lender has monopoly power over the borrower and could raise profits by setting $F_I = p$. But $R < p$ is not possible in equilibrium, since all rates would have to be less than p and the initial lender discount expected profits as first period would be negative (see (9)). ■

Proof of Proposition 1.

As shown in Lemma 3, it is without loss to focus attention on contracts with L that is fully utilized, i.e., such that if given this contract the consumer borrows L. We use the following definition:

Definition 1 *A contract $(F, R, L) \in \Theta$ is nondegenerate if 1) $L > 0, R < 1/\rho$, and 2) assuming the consumer accepts such a contract in first period (given equilibrium policies), we have $L = b_1 > 0$.*

We assume $B > 0$. This implies that the consumer borrows on zero profit contract $F = R = p$ in both periods. Since it is trivial, we omit the proof.

Assumption 1 $B > 0$.

Withough a loss, we restrict the contract space to a compact set:

Assumption 2 *Contract space is compact and given by*

$$\Theta = \{(F, R, L\} : 0 \leq F \leq R \leq 1/\rho, 0 \leq L \leq 3\bar{Y}\}. \tag{24}$$

It is clear that the consumer cannot repay more than her maximum lifetime income discounted at zero rate, hence the bound on L. It is also clear that interest rate R that exceed $\frac{1}{\rho}$ results in no borrowing and hence without loss we can impose such a bound. Same for $F \leq 1$. We impose some arbitrarily low but finite lower lower bound on F. (In quantitative and numerical analysis of the model this bound is at zero.)

We are interested in characterizing the optimal contract extended by the first period lender. This contract solves:

$$\max_{(F,R,L)\in\Theta} U(F, R, L) \text{ s.t. } \Pi(F, R, L) = 0, \tag{25}$$

as implied by (3) after plugging in $\eta = 1$. The following sequence of lemmas provides a preliminary characterization of the objects involved in this optimization problem, that is, U and Π, and show it is well-defined (max exists).

Lemma 5 *Let (F, R, L) be a nondegenerate contract. Then, the consumer borrows both in the first period and in the second (penultimate) period, i.e., $b_1 > 0$, $b_2 > 0$.*

Proof of Lemma 5. Profit in the first period is $(F-p)b_1^+$, where $b_1^+ := \max\{0, b_1\}$. Contract is nondegenerate and hence $b_1^+ > 0$ by assumption (L can be assume tight, see Lemma 1). If $)F < p$, the first period lender incurs a loss in the first period that must be recouped in the second period, which in turn requires that the consumer borrows in the second period. Specifically, second period profit is $\rho R b_2^+$, if the consumer refinances, and $(R-p)b_2^+$, if the consumer does not refinance. Hence, we must have $b_2^+ > 0$ in either case. If, on the other hand, $F = p$, by the zero profit condition we must have $R = p$. It is easy to verify that for $B > 0$ the consumer must borrow in the second period's persistent high state (no refinancing in this case). ■

The above lemma implies that we can ignore discontinuity that occurs when b_1 and b_2 switch sign, by assuming $b_1 > 0, b_2 > 0$ throughout, which we do from now on.

Lemma 6 *Let $(F, R, L) \in \Theta$ be a nondegenerate equilibrium contract that solves (25) with $R < 1/\rho$. Then, 1) first period lender's expected profit as a function of contract terms and consumer borrowing is given by*

$$\tilde{\Pi}(F, R, L; b_1, b_2) := \pi_1(F, b_1) + (1-p)\pi_2(R, b_2),$$

where

$$\begin{aligned} \pi_1(F, b_1) &:= (F-p)b_1^+ \\ \pi_2(R, b_2) &:= (\lambda_R \rho R + (1-\lambda_R)(R-p))b_2^+ \end{aligned} \tag{26}$$

and

$$\lambda_R := \begin{cases} 1 \text{ if } R > \frac{p}{1-\rho} \\ 0 \text{ if } R \leq \frac{p}{1-\rho}, \end{cases} \tag{27}$$

The profit function is continuous with respect to F, R, L and b_1, b_2 and,as long as $b_1 > 0, b_2 > 0$, it is continuously differentiable function except where λ switches value. 2) The relevant part of consumer indirect function that determines b_1, b_2 is given by

$$U(F, R, L) := \max_{b_1 \leq L, b_2} \tilde{U}(F, R, L; b_1, b_2), \tag{28}$$

where

$$\tilde{U}(F, R, L; b_1, b_2) := u_1(c_1(F, b_1)) + \beta(1-p)u_2(c_2(R, b_1, b_2)) + \beta^2(1-p)^2 u_3(c_3(b_2)).$$

and

$$\begin{aligned} c_1(F, b_1) &:= \bar{Y} - B + b_1 - F b_1^+ \\ c_2(R, b_1, b_2) &:= \bar{Y} - b_1 + b_2 - (\lambda_R(\rho R + p) + (1-\lambda_R)R)b_2^+ \\ c_3(b_2) &:= \bar{Y} - b_2. \end{aligned}$$

(Note that U ignores continuation utility following default state, i.e., low income state, which is irrelevant since it does not depend on debt and permanently severs the relation with the lender.) For any fixed nondegenerate tight contract the consumer problem has a unique solution and both the indirect utility function U and policy functions b_1, b_2 are continuous functions of contract terms F, R, L. 3) Furthermore, for $b_1 > 0, b_2 > 0$, as ensured by Lemma 5, and $R \neq p/(1-\rho)$, the envelope theorem implies

$$U_F = u_1' b_1 \tag{29}$$
$$U_R = \beta(1-p)u_2'(\lambda_R \rho + (1-\lambda_R))b_2, \tag{30}$$

where u_t' corresponds to $u'(c_t)$, and when borrowing constraint in the first period binds, additionally

$$U_L = u_1'(1-F) \; - \beta(1-p)u_2' > 0, \tag{31}$$

where b_1, b_2 are optimal policies that solve the consumer problem. (At $R = \frac{p}{1-\rho}$, the problem is not differentiable because λ_R switches sign.) 4) The profit function $\tilde{\Pi}$ that assumes optimal consumer policy function defines Π in (25) and it is a continuous function of contract terms F, R, L.

Proof of Lemma 6.

Part 1: We use results from Lemma 1 and Lemma 3 to recast (10) and (11). The continuity of the profit function follows from the fact that, at $R = \frac{p}{1-\rho}$, which is the point the consumer refinances in the second period, we have:

$$R - p = \frac{p}{1-\rho} - p = \frac{p-(1-\rho)p}{1-\rho} = \frac{\rho p}{1-\rho} = \rho R, \tag{32}$$

which is the only point suspect of discontinuity. The same argument applies to the consumer budget constraint, which is also continuous at the refinancing threshold.

Part 2: By Lemma 5, the consumer borrows in both periods and so $b_1 > 0, b_2 > 0$. Borrowing constraint in the second period is nonbinding, as shown in 1) above. Continuity of $c_2(R, b_2)$ with respect to R at $R = \frac{p}{1-\rho}$ similarly follows from (32) above. Accordingly, the consumer problem involves a standard maximization of a concave utility function subject to a set of linear equality and inequality constraints that together define a convex set (we assume parameters are such that for at least L=0 it is a nonempty problem). Hence, policy functions are well-defined and continuous with respect to R, F, L by the maximum theorem for convex programming.

Part 3: The envelope theorem for constrained optimizations applies everywhere except for $R = p/(1-\rho)$, where λ_R switches value and implies a kink. The envelope condition is

$$\begin{aligned}
\frac{dU}{dF} &= U_F := \frac{\partial U}{\partial F} = u_1' b_1 \\
\frac{dU}{dR} &= U_R := \frac{\partial U}{\partial R} = \beta(1-p)u_2'(\lambda_R \rho + (1-\lambda_R))b_2 \\
\frac{U_F}{U_R} &= \frac{u_1' b_1}{\beta(1-p)u_2'(\lambda_R \rho + (1-\lambda_R))b_2} = \frac{u_1' \pi_{1F}}{\beta(1-p)u_2'(1-p)\pi_{2R}},
\end{aligned}$$

given $b_1 > 0, b_2 > 0$ by Lemma 5. The above conditions also hold when the borrowing constraint binds in the first period, i.e. $b_1 = L$ and it is binding. This is because in such a case policy function simply does not change because the constraint binds. Since c_2 is continuous at $R = \frac{p}{1-\rho}$. (Part 4 is clear given part 3.) ■

Montonone transformation of the contract space.

Before we proceed, we show that the where the consumer's budget constraint and profit function $\tilde{\Pi}$ are nondifferentiable, namely at $R = p/(1-\rho)$, where λ_R switches sign, has an identical kink that can be, in effect, "straightened" by appropriately redefining the contract space. This will allow us to ensure global differentiability of policy functions and value functions.

Consider the following transformation:

$$R(\hat{R}) = \frac{1}{\rho}\max(\hat{R} - \frac{p}{1-\rho}, 0) + \min(\hat{R}, \frac{p}{1-\rho}) \tag{33}$$

It is clear that $R(\hat{R})$ is a strictly increasing, positive valued, and continuous function of $\hat{R}$, with $R(p) = p$, $R(p/(1-\rho)) = p/(1-\rho)$, and hence the formula of $\lambda_{\hat{R}}$ is the same. Moreover, we can bound $\hat{R}$ to be consistent with Θ by assuming $0 \le \hat{R} \le 1$, since $R(1) = 1/\rho$. Clearly, $R(0) = 0$. We now show that after redefining the contract space both the profit function $\tilde{\Pi}$ and the agent's budget constraint are continuously differentiable functions of $\hat{R}$. To that end, note that, if $\hat{R} < \frac{p}{1-\rho}$, we have $c_2(R, b_1, b_2) := \bar{Y} - b_1 + b_2 - \hat{R}b_2^+$. Similarly, if, $\hat{R} > \frac{p}{1-\rho}$, we have $c_2(R, b_1, b_2) := \bar{Y} - b_1 + b_2 - \hat{R}b_2^+$, as implied by the evaluation of the min operators. Similarly, given the formula for λ_R in (27), and $\tilde{\Pi}$, if $\hat{R} < \frac{p}{1-\rho}$, it is easy to verify that we have $\tilde{\Pi}(F,R,L;b_1,b_2) := (F-p)b_1^+ + (1-p)(\hat{R}-p)b_2^+$, and if $\hat{R} \geq \frac{p}{1-\rho}$, we have $\tilde{\Pi}(F,R,L;b_1,b_2) := (F-p)b_1^+ + (1-p)(\hat{R}-p)b_2^+$. Accordingly, while the original problem had a kink at refinancing point, $R = \frac{p}{1-\rho}$, the transformed problem does not. We will now work with $\hat{R}$ instead of R on the entire compact domain and ensure differentiability.

Next, we note the following technical properties of the the profit function $\Pi(F,\hat{R})$ and utility function $U(F,\hat{R})$:

Lemma 7 (Global differentiability of policy and value functions) *Let $(F,R,L) \in \Theta$ be a nondegenerate equilibrium contract that solves (25) with $R < 1/\rho$. $\tilde{\Pi}(F,R(\hat{R}),b_1,b_2)$, $\tilde{U}(F,R(\hat{R}),b_1,b_2)$ are globally differentiable with respect to $F,\hat{R},b_1,b_2$, where $R(\hat{R})$ is given by (33). Assuming $L = b_1 > 0$ is nonbinding, $\Pi(F,R(\hat{R}))$, $U(F,R\hat{R})$ and consumer policy functions $b_1(F,R(\hat{R}))$, $b_2(F,R(\hat{R}))$ are continuously differentiable with respect to $F,\hat{R}$. Furthermore, $\Pi(F,R(\hat{R}))$ is strictly increasing in F and $\hat{R}$ at $F = p = \hat{R}$, and $U(F,R(\hat{R}))$ is strictly decreasing in F and R if $b_1 > 0$ and $b_2 > 0$, respectively.*

Proof of Lemma 7. Note by Lemma 5 that $b_1 > 0, b_2 > 0$. The first part directly follows from applying transformation (33), as commented in text above. Differentiability of $\Pi(F,R(\hat{R}))$ and $U(F,R(\hat{R}))$ follows from the fact that $\tilde{\Pi} = (F-p)b_1(F,\hat{R}) + (1-p)(\hat{R}-p)b_2(F,\hat{R})$ is continuously differentiable, similarly for the utility function, and consumer policy functions $b_1(F,\hat{R})$, $b_2(F,\hat{R})$ are continuously differentiable wrt F,R (L nonbinding). The latter property is implied by the fact that Euler equations C_1 and C_2 are necessary and sufficient to pin down policy functions b_1,b_2. Since utility function is continuously differentiable, policy functions are continuously differentiable with respect to contract terms F,R by the implicit function theorem. Implicit function theorem applies here by the following argument. Combine Euler equations C_1 and C_2 to replace C_2 by C_2' defined as follows:

$$C_2' : E_2' := u_1(1-F) + \frac{\beta^2(1+p)^2)}{1-\hat{R}}u_3 = 0.$$

Implicit function theorem requires that the gradient vectors ∇C_1, $\nabla E_2'$ with respect to b_1, b_2 be linearly independent. This is ensured by the fact that the partial derivative of left-hand side of ∇E_1 with respect to b_1 is exactly equal to the partial derivative of left-hand side of $\nabla E_2'$ with respect to b_1, and nonzero by assumption Assumption 2, while partials with respect to b_2 have an opposite sign and are both nonzero:

$$\nabla E_1 = \begin{bmatrix} \frac{\partial E_1}{\partial b_1} \\ \frac{\partial E_1}{\partial b_2} \end{bmatrix} = \begin{bmatrix} u_1''(1-F)^2 + \beta(1-p)u_2'' \\ -\beta(1-p)(1-\hat{R})u_2'' \end{bmatrix} = \begin{bmatrix} -^* \\ + \end{bmatrix},$$

$$\nabla E_2' = \begin{bmatrix} \frac{\partial E_2'}{\partial b_1} \\ \frac{\partial E_2'}{\partial b_2} \end{bmatrix} = \begin{bmatrix} u_1''(1-F)^2 + \beta(1-p)u_2'' \\ \frac{\beta^2(1-p)^2}{(1-\hat{R})}u_3'' \end{bmatrix} = \begin{bmatrix} -^* \\ - \end{bmatrix},$$

where $+$ ($-$) means the expression is strictly positive (negative) and * means equal value. The second part follows from differentiability of the profit function at $F = p = \hat{R}$ and the fact that profit is zero for $F = p = \hat{R}$ and strictly negative when either $F < p, \hat{R} = p$ or $\hat{R} < p, F = p$ (recall that b_1, b_2 are both positive). ■

We now transform the contracting problem to a more tractable version for characterization.

Lemma 8 (Transformed contracting problem) *Any nondegenerate contract $(F,R(\hat{R}),L) \in \Theta$ with $R < 1/\rho$ solves (25) if and only if it solves:*

$$\max_{(F,\hat{R},L)\in\hat{\Theta},b_1\leq L,b_2} U(F,\hat{R},L) \tag{34}$$

subject to

$$\begin{aligned}
C_1 &: \; E_1 := u'(c_1(F,b_1))(1-F) - \beta(1-p)u'(c_2(R(\hat{R}),b_1,b_2)) \geq 0 && (35)\\
C_2 &: \; E_2 := u'(c_2(R(\hat{R}),b_1,b_2))(1-\hat{R}) - \beta(1-p)u'(c_3(b_2))) = 0 && (36)\\
C_3 &: \; \tilde{\Pi}(F,R(\hat{R}),b_1,b_2) := (\pi_1(F,b_1) + (1-p)\pi_2(R(\hat{R}),b_2)) \geq 0. && (37)
\end{aligned}$$

where

$$\hat{\Theta} = \{(F,R,L) : -\underline{F} \leq F \leq 1, 0 \leq \hat{R} \leq 1, 0 \leq L \leq 3\bar{Y}\}, \tag{38}$$

and $c_1, c_2, c_3, U, \tilde{\Pi}, \lambda_R$ *as defined in Lemma 6 after plugging* $R(\hat{R})$ *given by* (33).

Proof. For the first part, note the transformed problem uses (33) and extends the contracting problem to incorporate consumer's first order conditions and choice variables b_1, b_2. When credit constraint in the first period is nonbinding, consumer's first order conditions are

$$\begin{aligned}
u_1'(1-F) &= \beta(1-p)u_2'\\
u_2'(1-\hat{R}) &= \beta(1-p)u_3'.
\end{aligned}$$

If L is binding, the first equation turns to inequality:

$$u_1'(1-F) > \beta(1-p)u_2'.$$

This corresponds to C_1, C_2, and the constraint $b_1 \leq L$. By Lemma 6, consumer's first order conditions are both necessary and sufficient for a nondegenerate equilibrium contract (with $\hat{R} \leq 1$), since it is a standard convex programming defined a compact space with a set of linear constraints. Hence the max is well-defined and the two problems are trivially equivalent. ■

Using the above results, we now return to the proof of Propositions 1.

Claim 1: Assume $L > 0$ is nonbinding in first period in a positive credit economy ($b_1 > 0$), i.e., $U_L(F,\hat{R},L) = 0$, then (12) holds, $F = p = R$ and L slack.

Using Lemma 8, we combine constraints C_1 and C_2, and divide both sides by $(1-\hat{R})$, to replace C_2 with

$$C_2' : E_2' := u'(c_1(F,b_1))(1-F) - \frac{(\beta(1-p))^2 u'(c_3(b_2))}{1-\hat{R}} = 0.$$

This is possible in this case since C_1 is assumed to hold with equality. We consider the following Lagrangian corresponding to (34) under the above assumption:

$$\begin{aligned}
\mathcal{L} \quad = \quad & U(F,\hat{R},b_1) + \\
C_3 : \quad & \mu(\pi_1(F,b_1) + (1-p)\pi_2(\hat{R},b_2)) + \\
C_1 : \quad & \gamma\left(u'(c_1(F,b_1))(1-F) - \beta(1-p)u'(c_2(\hat{R},b_1,b_2))\right) + \\
C_2' : \quad & \xi\left(u'(c_1(F,b_1))(1-F) - \frac{(\beta(1-p))^2 u'(c_3(b_2)}{1-\hat{R}}\right).
\end{aligned}$$

Note that the objective function directly assumes $b_1 = L$ and hence we omit the first period borrowing constraint (recall that fully utilized credit line can be assumed without loss).

Karush-Kuhn-Tucker (KKT) conditions comprise *stationarity conditions*:

$$
\begin{aligned}
F &: \quad U_F + \mu\pi_{1F} - \gamma\left(u_1''(1-F)b_1 + u_1'\right) - \xi\left(u_1''(1-F)b_1 + u_1'\right) = 0 && (39)\\
\hat{R} &: \quad U_R + \mu(1-p)\pi_{2\hat{R}} + \gamma\left(\beta(1-p)u_2''b_2\right) \ + \xi\frac{(\beta(1-p))^2 u_3''}{(1-\hat{R})^2} = 0 && (40)\\
b_1 &: \quad U_{b_1} + \gamma\left(u_1''(1-F)^2 + \beta(1-p)u_2''\right) + \xi\left(u_1''(1-F)^2\right) = -\mu\pi_{1b_1} && (41)\\
b_2 &: \quad -\gamma\beta(1-p)u_2'' + \xi\frac{(\beta(1-p))^2 u_3''}{1-\hat{R}} = -\mu(1-p)\pi_{2b_2}, && (42)
\end{aligned}
$$

dual feasibility conditions : $\mu \geq 0,\ \gamma \geq 0,$ *complementarity slackness conditions* :

$$
\begin{aligned}
\mu\left(\pi_1(F, b_1) + (1-p)\pi_2(\hat{R}, b_2)\right) &= 0\\
\gamma\left(u'(c_1(F, b_1))(1-F) - \beta(1-p)u'(c_2(\hat{R}, b_1, b_2))\right) &= 0,
\end{aligned}
$$

and *primal feasibility conditions* given by C_3, C_1 and C_2'. If constraint qualification (CQ) holds globally, which we verify later, these conditions are sufficient for an extremum for an the interior extremum on Θ. It is clear that boundary point $R = 1/\rho$ ($\hat{R} = 1$) cannot be optimal, since then $F = p$ by zero profit condition and $F = p = R$, $L > 0$ nonbinding is a strictly better feasible contract since it does not shut down borrowing in the second period. The boundary point $F = p = R$, $L >$ nonbinding is what we are about to show.

From (41), we infer $\xi \leq 0$, since otherwise the left-hand side of (41) would have contradicted the fact that $-\mu\pi_{1b_1} \geq 0$, given $\gamma \geq 0, \mu \geq 0, U_{b_1} = 0$ and

$$
\begin{aligned}
\gamma(u_1''(1-F)^2 + \beta(1-p)u_2'') &< 0 && (43)\\
u_1''(1-F) &< 0.
\end{aligned}
$$

Analogously, from (42), we infer $\xi \geq 0$, since otherwise the left-hand side of (42) would have contradicted the fact that $-\mu(1-p)\pi_{2b_2} \leq 0$, given $\gamma \geq 0, \mu \geq 0$, and

$$
\begin{aligned}
-\gamma\beta(1-p)u_2'' &> 0,\\
\frac{(\beta(1-p))^2 u_3''}{1-\hat{R}} &< 0.
\end{aligned}
$$

Concluding, we have $\xi = 0$ and hence the first order condition with respect to b_2 in (42) reduces to

$$
b_2 : -\gamma\beta(1-p)u_2''(1-\hat{R}) = -\mu(1-p)\pi_{2b_2}.
$$

Since $-\mu(1-p)\pi_{2b_2} \leq 0$, and $\beta(1-p)u_2'' < 0$, we must also have $\gamma = 0$. ($\gamma > 0$ contradicts the fact that the right-hand side of the first order condition with respect to b_2 is negative or zero and $\gamma \geq 0$.)

Given $\gamma = \xi = 0$, KKT stationary condition reduces to:

$$
\begin{aligned}
U_F + \mu\pi_{1F} &= 0 && (44)\\
U_{\hat{R}} + \mu(1-p)\pi_{2\hat{R}} &= 0 && (45)\\
-\mu\pi_{1b_1} &= 0 && (46)\\
-\mu(1-p)\pi_{2b_2} &= 0. && (47)
\end{aligned}
$$

Note that $\mu = 0$implies $U_F = U_{\hat{R}} = 0$, which is not possible (the consumer borrows a positive amount and a change in interest rates strictly shrinks or expands a binding budget constraint under local non-satiation), and hence $\mu > 0$. Equations (44) and (45) then imply

$$\frac{U_F}{U_{\hat{R}}} = \frac{\pi_{1F}}{(1-p)\pi_{2\hat{R}}}$$

Using the envelope condition in (29-30), after plugging in $R(\hat{R})$ from (33) (which does not change the formula), we obtain

$$\frac{u_1'\pi_{1F}}{\beta(1-p)u_2'\pi_{2\hat{R}}} = \frac{U_F}{U_{\hat{R}}} = \frac{\pi_{1F}}{(1-p)\pi_{2\hat{R}}},$$

and

$$u_1' = \beta u_2'.$$

The above implies that (12) holds when KKT conditions are satisfied. For KKT to be satisfied, recall that C_1 must hold with equality, which requires:

$$(1-F)u_1' - \beta(1-p)u_2' = 0,$$

which requires $1-F = p$. Furthermore, since $\mu > 0$, the only way to satisfy (46) and (47) is $\pi_{1b_1} = F-p = 0$, $\pi_{2b_2} = \hat{R} - p = 0$, which implies $F = p$, $\hat{R} = p$, and thus $R = p$. We conclude that $F = p, R = p$ is the only contract that satisfies KKT conditions and it is better than boundary contract $R = 1/\rho$, as noted in the beginning. We now verify **constraint qualification (CQ)** to show that this is the only candidate interior extremum.

$^{\$}$To verify CQ, we will use Mangasarian-Fromovitz constraint qualification criterion (MFCQ). Accordingly, i) we must find a vector $z \in R^4$ such that $\nabla E_1^T z = 0$, $\nabla E_2'^T z = 0$ and $\nabla \tilde{\Pi}^T z = 0$; and ii) establish that the gradients corresponding to all equality constraints, i.e., ∇E_1 and $\nabla E_2'$, are linearly independent. Part ii has already been done in the proof of Lemma 7, which established linear independence of the last two coordinates. It thus remains to verify part i. To that end, consider two cases $F < p$ and $F > p$, which by zero profit condition implies $R < p$ or $R > p$. The gradients are:

$$\nabla E_1 = \begin{bmatrix} \frac{\partial E_1}{\partial F} \\ \frac{\partial E_1}{\partial R} \\ \frac{\partial E_1}{\partial b_1} \\ \frac{\partial E_1}{\partial b_2} \end{bmatrix} = \begin{bmatrix} -u_1''(1-F) - u_1' \\ \beta(1-p)u_2'' \\ u_1''(1-F)^2 + \beta(1-p)u_2'' \\ -\beta(1-p)(1-\hat{R})u_2'' \end{bmatrix} = \begin{bmatrix} ? \\ - \\ - \\ + \end{bmatrix}, \tag{48}$$

$$\nabla E_2' = \begin{bmatrix} \frac{\partial E_2'}{\partial F} \\ \frac{\partial E_2'}{\partial R} \\ \frac{\partial E_2'}{\partial b_1} \\ \frac{\partial E_2'}{\partial b_2} \end{bmatrix} = \begin{bmatrix} -u_1''(1-F) - u_1' \\ -\frac{\beta^2(1-p)^2 u_3''}{(1-\hat{R})^2} \\ u_1''(1-F)^2 \\ \frac{\beta^2(1-p)^2}{(1-\hat{R})} u_3'' \end{bmatrix} = \begin{bmatrix} ? \\ + \\ - \\ - \end{bmatrix}, \tag{49}$$

$$\nabla \tilde{\Pi} = \begin{bmatrix} \frac{\partial \tilde{\Pi}}{\partial F} \\ \frac{\partial \tilde{\Pi}}{\partial R} \\ \frac{\partial \tilde{\Pi}}{\partial b_1} \\ \frac{\partial \tilde{\Pi}}{\partial b_2} \end{bmatrix} = \begin{bmatrix} b_1 \\ (1-p)b_2 \\ F-p \\ (1-p)(\hat{R}-p) \end{bmatrix} = \begin{bmatrix} + \\ + \\ -^* \\ +^* \end{bmatrix}, \tag{50}$$

where "+ (−)" means the sign of the partial derivative is strictly positive (negative), "*" denotes weak sign, and "?" means the sign is ambiguous.

Consider $z^T = (0, -1, 0, -(1-\hat{R})^{-1})$. It is straightforward to verify that z zeros out the sum of the second and the third coordinate of the gradient vectors ∇E_1 and $\nabla E_2'$, since they are in proportion $(1-\hat{R})^{-1}$ to one another. Furthermore, each coordinate of z is negative and it multiplies only positive elements of vector $\nabla \tilde{\Pi}$, yielding a negative product as required by $\nabla \tilde{\Pi}^T z < 0$. Finally, for contract $F = p = \hat{R}$, we use the linear independence CQ criterion by noting that all three gradient vectors are linearly independent. Note that the third and fourth coordinate of $\nabla\tilde{\Pi}$ are both zero. Since it is not possible to generate a zero vector from the last two coordinates of $\nabla E_2'$ "$(-,-)$", ∇E_1 "$(-,+)$" by multiplying them by scalars that are not all zero, the vectors are linear independent given our earlier observation that $\nabla E_2'$, ∇E_1 are mutually linearly independent (see proof of Lemma 7).

We have identified all contracts that meet KKT sufficiency condition for an extremum on the interior on the contract space and excluded the boundary as candidates for global maximum. By extreme value theorem, we know that a continuous function on a compact set must attain a global maximum and global minimum. Since we can rank them, the contract $F = p = R$ is a global maximum among contracts with a nonbinding L.

Claim 2: Assume $L > 0$ is binding in a positive credit economy and $F < p, R > p$. Then, $u_1' < \beta u_2$.

We will show that, unless $u_1' \leq \beta u_2$, there exists a profitable deviation that raises borrower utility, while leaving profits unchanged, a contradiction.

To that end, we note the following: 1) Profit from the first period borrowing is $\pi_1(L) := (F(L) - p)L$. Accordingly, to maintain the constant period profit flow from the first period following a marginal increase of L by some dL while simultaneously modifying F by some dF requires that dF be given by:

$$\begin{aligned}
\frac{d\pi_1(L)}{dL} &= \frac{dF(L)}{dL}L + F(L) - p = 0 \\
dF(L) &= \frac{p - F(L)}{L}dL.
\end{aligned}$$

2) Total profit from both periods is $\Pi = \pi_1 + (1-p)\pi_2$, and it only increases following such a variation. Note that an increase in L, by raising b_1 by dL, trivially implies higher b_2 by some $db_2 > 0$ by the second period Euler equation

$$u_2'(Y - b_1 + b_2(1-R))(1-R) = \beta(1-p)u_3'(Y - b_2). \tag{51}$$

But higher b_2 raises profits because $\pi_2 = (\hat{R} - p)b_2$ and by assumption $\hat{R} \geq p$. 3) Assuming a binding borrowing constraint in the first period, consumption in first period is $c_1 = \bar{Y} - B + L - LF(L)$ and hence by 1) above we have

$$dc_1(L) = dL - L\frac{p - F(L)}{L}dL - F(L)dL = (1-p)dL,$$

which, by the envelope theorem, implies that the total induced change of lifetime utility is

$$dU = u_1'dc_1(L) - \beta(1-p)u_2'dL = dL(1-p)(u_1' - \beta u_2').$$

(We use envelope theorem to drop terms that characterize the impact of b_2 induced indirectly by the change in b_1. Since b_2 is set optimally by the agent in second period envelope theorem applies and such an indirect effect is nil.) Note that 3) finishes the proof because either $u_1' - \beta u_2' < 0$ or we have a contradicted the hypothesis. Note here that profits in the second period π_2 actually strictly go up when constraint is relaxed, implying the lender could cut $\hat{R}$ by some amount. Hence it must be that a strict equality is requires for the deviation not to increase the value of the program.

Claim 3: Assume $L > 0$ is binding and $F < p, R > p$ in a positive credit equilibrium ($b_1 > 0$) and LMC is satisfied. Then, $u_1' > \beta u_2$, which contradicts Claim 2 and proves L must be nonbinding.

We first solve KKT conditions assuming binding borrowing constraint in first period and show CQ holds globally, implying KKT conditions are sufficient for an extremum on the interior of Θ. The boundary zero profit contracts: 1) $F = p = R$ or 2) $F = p, R = 1/\rho$, could give rise to a corner solution in a positive credit equilibrium with a binding L. However, for $F = p = R$, since $\tilde{\Pi} \equiv 0$, relaxing the borrowing constraint by raising L unambiguously increases the borrower utility and leaves profits unchanged, a contradiction. $R = 1/\rho$, on the other hand, is worse than $F = p = R$, L slack, as noted in the proof of Claim 1.

We begin by showing KKT conditions imply: $u_1' > \beta u_2'$. Using Lemma 8, consider the Lagrangian

$$\begin{aligned}
\mathcal{L} \quad &= \quad U(F, \hat{R}, b_1) + \\
C_3: \quad & \mu(\pi_1(F, b_1) + (1-p)\pi_2(\hat{R}, b_2)) + \\
C_1: \quad & \gamma\left(u'(c_1(F, b_1))(1-F) - \beta(1-p)u'(c_2(\hat{R}, b_1, b_2))\right) + \\
C_2: \quad & \xi\left(u'(c_2(\hat{R}, b_1, b_2))(1-\hat{R}) - \beta(1-p)u'(c_3(b_2))\right).
\end{aligned} \tag{52}$$

(We are not allowed to combine the constraints as we did in proof of Claim 1. Note that the objective function assumes $b_1 = L$ and hence we omit the borrowing constraint.)

Karush-Kuhn-Tucker (KKT) conditions comprise:

$$F \;:\; U_F + \mu\pi_{1F} - \gamma\left(u_1''(1-F)b_1 + u_1'\right) = 0 \tag{53}$$

$$R \;:\; U_{\hat{R}} + \mu(1-p)\pi_{2R} + \gamma\left(\beta(1-p)u_2''b_2\right) + \xi\left(-u_2''b_2^+(1-\hat{R}) - u_2'\right) = 0 \tag{54}$$

$$b_1 \;:\; U_{b_1} + \gamma\left(u_1''(1-F)^2 + \beta(1-p)u_2''\right) - \xi\left(u_2''(1-\hat{R})\right) = -\mu\pi_{1b_1} \tag{55}$$

$$b_2 \;:\; -\gamma\left(\beta(1-p)u_2''(1-\hat{R})\right) + \xi\left(u_2''(1-\hat{R})^2 + \beta(1-p)u_3''\right) = -\mu(1-p)\pi_{2b_2}, \tag{56}$$

$$\mu \geq 0, \;\; \gamma \geq 0 \tag{57}$$

$$\begin{aligned}
\mu\left(\pi_1(F, b_1) + (1-p)\pi_2(\hat{R}, b_2)\right) \quad &= \quad 0 \\
\gamma\left(u'(c_1(F, b_1))(1-F) - \beta(1-p)u'(c_2(\hat{R}, b_1, b_2))\right) \quad &= \quad 0,
\end{aligned} \tag{58}$$

and constraints C_1, C_2 and C_3.

Complementarity slackness condition (58) readily implies $\gamma = 0$ and we drop C_1 from the set of active constraints. By the envelope condition in (31), and the assumption underlying this case, we know

$$U_{b_1} = u_1'(1-F) - \beta(1-p)u_2' > 0. \tag{59}$$

KKT stationarity conditions thus boil down to:

$$F \;:\; U_F + \mu\pi_{1F} = 0 \tag{60}$$

$$R \;:\; U_{\hat{R}} + \mu(1-p)\pi_{2\hat{R}} + \xi\left(-u_2''b_2^+(1-\hat{R}) - u_2'\right) = 0 \tag{61}$$

$$b_1 \;:\; U_{b_1} - \xi\left(u_2''(1-\hat{R})\right) = -\mu\pi_{1b_1} \tag{62}$$

$$b_2 \;:\; \xi\left(u_2''(1-\hat{R})^2 + \beta(1-p)u_3''\right) = -\mu(1-p)\pi_{2b_2}, \tag{63}$$

Since $\hat{R} \geq p$ on Θ, (63) implies $\xi \geq 0$ because $-\mu(1-p)\pi_{2b_2} \leq 0$ (recall that$\mu \geq 0$ by (57)). If $\xi = 0$, (62)-(63) imply $\pi_{1b_1} = (F-p) < 0$ and $\pi_{2b_2} = (\hat{R}-p) = 0$, which implies negative profits, a contradiction. Hence, $\xi > 0$.

By the envelope condition (29) and (30), equation (60) gives $u_1' = \mu$. Substituting U_R from envelope condition (30) and μ, equation (61) gives

$$R : -\beta u_2'(1-p)\pi_{2R} + u_1'(1-p)\pi_{2R} + \xi(...) = 0. \tag{64}$$

Since $\pi_{2R} = \lambda\rho + (1-\lambda)b_2^+ > 0$, (61) now boils down to

$$R : u_1' + \xi\left(-u_2'' b_2^+(1-\hat{R}) - u_2'\right)/(1-p) = \beta u_2'. \tag{65}$$

The expression in bracket, i.e., $-u_2'' b_2^+(1-\hat{R}) - u_2'$, is strictly negative when the monotonicity criterion is satisfied.[32]

Equation (65) is thus of the form:

$$u_1' + (\text{“strictly negative term”}) = \beta u_2', \tag{66}$$

and hence

$$u_1' \geq \beta u_2', \tag{67}$$

which finishes the first part of the proof.

We finish by showing that CQ holds globally . Note that we have two active constraints: E_2 in (36) and E_3 in (37). It suffices to show that these two constraints are linearly independent. To that end, we note that the first coordinate of the gradient vector ∇E_2 (derivative with respect to F) is zero and the first coordinate of the gradient vector ∇E_3 is strictly positive (since $b_1^+ > 0$). It is now clear that the two gradients are trivially linearly independent because it is not possible to generate a zero by multiplying a positive number by a nonzero scalar. ■

Proof of the first part of Proposition 2.

The preliminary lemmas in the proof of Proposition 1 readily generalize with one caveat: The consumer may incorrectly anticipate ex ante that once she becomes a patient type next period ($\eta = 1$) incumbent lender will reprice the line by lower rate ex post to some $\underline{R} < R$. From the lender's perspective nothing changes on the equilibrium path, since consumer's type never changes from $\eta < 1$ to $\eta = 1$, but this off equilibrium path expectation may affect consumer's ex ante behavior. As before, repricing on the equilibrium path does not occur for the same reason: The initial lender would have lowered R to begin with in the first period, which we can assume without a loss. For now we assume $\underline{R} = R$ (also in derivatives) and at the end we show that it is indeed the relevant case (see * at the end). We similarly use envelope conditions and Lemma 6, which applies here with basic modifications. The enveloped condition in this case is:

$$\frac{U_F}{U_{\hat{R}}} = \frac{u_1' b_1}{\beta(1-p)u_2' b_2} = \frac{u_1' \pi_{1F}}{\beta(1-p)u_2' \pi_{2\hat{R}}} \frac{b_2^\eta}{b_2}, \tag{68}$$

where, recall, now we have $\frac{b_2^\eta}{b_2} > 1$ for $\eta < 1$, and which also holds when borrowing constraint in first period binds) We omit the deriviation as it is straightforward. With $\underline{R} = R$, the problem is similar to the one used in the proof of Proposition 1. We similarly apply the transformation $R(\hat{R})$ given by (33) to obtain ex ante consumer problem:

$$U^\eta(F, \hat{R}, L) := \max_{b_1, b_2} u_1(c_1) + \beta\eta(1-p)[u_2(c_2) + \beta(1-p)u_3(c_3)],$$

subject to $c_1 = \bar{Y} - B + b_1 - Fb_1^+$, $c_2 = \bar{Y} - b_1 + b_2 - \hat{R}b_2^+$, $c_3 = \bar{Y} - b_2$, and lender profit function

$$\tilde{\Pi}^\eta(F, R, L, b_1, b_2^\eta) = (F-p)b_1^+ + (1-p)(\hat{R}-p)b_2^{\eta+}$$

[32] To see this, note that this equation corresponds to the derivative of the second period Euler equation underlying (8), evaluated a a point where consumer Euler equation holds (C_2 applies). By the way of contradiction, note that if the sign of this derivative was positive, borrowing b_2^+ would have to fall for a fixed b_1 following an increase in interest rate $\hat{R}$ (and R) to restore the Euler equation by raising the marginal utility on the left-hand side and lowering it on the right-hand side (*ceteris paribus*). But this contradicts the monotonicity criterion.

, where b^{η} is ex post borrowing and solves

$$b_2^{\eta} = \max_{\hat{b}_2}[u(c_2(R, b_1, \hat{b}_2) + \eta\beta u(c_3(\hat{b}_2)],$$

instead of $b_2 = \max_{\hat{b}_2}[u(c_2(R, b_1, \hat{b}_2) + \beta u(c_3(\hat{b}_2)]$. It is clear from the above two problems that, for any fixed b_1, which as of the second period is indeed fixed, $b^{\eta} > b_2$.

Assume, by the way of contradiction, that the optimal contract is $F = p = \hat{R}$ $(R = p)$, with nonbinding $L > 0$. We can reject similarly $R = 1/\rho$, as we have done int he proof of Proposition 1. We will now show that, if $\eta < 1$, the initial lender has an incentive to deviate from this contract by lowering F and raising R along the zero profit line, while keeping credit constraint relaxed. By extreme value theorem, it thus must be that $F < p$ given Θ, and hence $R > p$ by the zero profit condition.

Consider two economies: the first economy, referred to as the *baseline economy*, is the economy with with $\eta < 1$ that this proof pertains to. The second economy, referred to as *counterfactual economy*, is an analogous economy with $\eta < 1$ except that borrowers discount factor in the second period does not change ex post and lenders know it. That is, the consumer is time-consistent, only discounts her future utility just like the hyperbolic consumer does ex ante. Using our notation, in the baseline economy profit function is

$$\Pi^{\eta}(F^{\eta}, \hat{R}) := (F - p)b_1(F, \hat{R}) + (1 - p)(\hat{R} - p)b_2^{\eta}(F, \hat{R}),$$

where $b_2^{\eta}(F, \hat{R})$ is the borrower's ex post policy function. In contrast, in the counterfactual economy, whose variables we superscript by c, it is

$$\Pi_c^{\eta}(F^{\eta}, \hat{R}) := (F - p)b_{1c}(F, \hat{R}) + (1 - p)(\hat{R} - p)b_{2c}(F, \hat{R}),$$

where $b_{2c}(F, \hat{R})$ is the borrower's ex ante policy function. In the counterfactual economy we can be sure that the borrower has rational expectations since she is time consistent (type never changes ex post and future self will discount at rate β).

It is clear that the proof of Proposition 1 applies to counterfactual economy without any changes. This can be verified by repeating each step. The monotonicity criterion is assumed for the ex ante problem and hence applies to the counterfactual economy. Consequently, the equilibrium contract is $F = p = R$, L slack.

Consider now a deviation from $\hat{R}$ by some $d\hat{R} > 0$ from $\hat{R} = p = F$ (L slack). By the implicit function theorem, the required offsetting change in dF^{η} to keep profits constant is

$$dF^{\eta}(\hat{R}) := -\frac{(1 - p)b_2^{\eta}(p, p)}{b_1(p, p)}d\hat{R}.$$

This can be calculated by implicitly differentiating the above profit functions at $F = p = \hat{R}$. It is clear that, since $dF^{\eta}(\hat{R}) < 0$ and $d\hat{R} > 0$, such a deviation is feasible. The borrowing constraint continues to be slack.

We will now use shorthand notation b_1 instead of writing $b_1(p, p)$ and similarly write $\frac{dF(\hat{R})}{d\hat{R}}$ instead of $\frac{dF^{\eta}(\hat{R})}{d\hat{R}}$. For any $\eta < 1$, note that

$$dF^{\eta}(\hat{R}) = \frac{(1 - p)b_2^{\eta}}{b_1}d\hat{R} < \frac{(1 - p)b_{c2}}{\bar{b}_1}d\hat{R} = dF_c^{\eta}(\hat{R}),$$

since, trivially, $b_2^{\eta} > b_{c2}$, and $b_1 = b_{c1}$. We also know that $U^{\eta} \equiv U_c^{\eta}$, since the ex ante problem of the borrower is identical across the two economies. The change in consumer utility from this deviation is thus given by

$$\begin{aligned} dU^{\eta} &= \frac{\partial U^{\eta}}{\partial F^{\eta}} dF^{\eta}(\hat{R}) + \frac{\partial U^{\eta}}{\partial \hat{R}^{\eta}} d\hat{R} \\ dU_c^{\eta} &= \frac{\partial U^{\eta}}{\partial F^{\eta}} dF_c^{\eta}(\hat{R}) + \frac{\partial U^{\eta}}{\partial \hat{R}^{\eta}} d\hat{R} \end{aligned}$$

which implies

$$dU^{\eta} - dU_c^{\eta} = \frac{\partial U^{\eta}}{\partial F^{\eta}}(dF^{\eta}(\hat{R}) - dF_c^{\eta}(\hat{R})) > 0,$$

since

$$\begin{aligned} dF^{\eta}(\hat{R}) &< dF_c^{\eta}(\hat{R}), \\ \frac{\partial U^{\eta}}{\partial F^{\eta}} &< 0. \end{aligned}$$

Concluding, we have shown that this deviation is feasible and raises consumer's ex ante utility without changing profits, a contradiction. To finish the proof consider now an alterantive deviation that could invalidate our argument: setting L binding and maintaining $F = p = R$. It is clear that this cannot be optimal: Consumer utility is lower and raising L is feasible for the lender and maintains the same level of profits, since $\tilde{\Pi} \equiv 0$ in this case. This proves the first part of the proposition. (By extreme value theorem we know that there exists a global maximum, and hence an optimal contract features $F < p$, $R > p$.)

*We now deal with the off equilibrium repricing issue that we assumed away in the beginning. By contradiction, suppose the optimal contract is: $F = p$, $R \geq p$, $\underline{R} < R$. This is the only possibility, since $F < p$ requires $R > p$, which is what we prove here. First note that the offer $F = p = R$ weakly undercuts any such contract and so we can consider it instead, possibly repriced down to some $\underline{R}$ off equilibrium.

Consider a deviation that marginally changes R ex ante by some $dR > 0$ such that $\underline{R} < R + dR$ to increase second period interest revenue and lowers F by some dF. Consumer's ex ante indirect utility must increase after this change, since she should rationally expect $\underline{R}$ and hence is insensitive to dR. At the same time, her first period utility unambiguously goes up because of a lower F (given $b_1 > 0$). Note that lender's ex post repricing policy must be independent of marginally different R, since the ex post post lender profit maximization problem is unaffected by the deviation and repricing is defined by profit maximization ex post. Accordingly, the consumer should not revise her expectations of repricing policy, as assumed. dR must also raise profits because we assumed in text that profits are increasing in R and F at contract $F = p = R$. This contradicts the hypothesis. ∎

Proof of second part of Proposition 2 and proof of Proposition 3.

We prove the second part of Proposition 2 showing that, for sufficiently high $0 < \eta\beta \leq 1$ and η sufficiently low, have $F = 0$ and refinancing in equilibrium. We combine it with the proof of Proposition 3 by using a general notation that introduces R_{-1} and nests both propositions. For proof of Proposition 2 assume $R_{-1} = 0$. We use KKT system to show that the solution has the stated comparative statics properties.

We consider a positive credit economy with L nonbinding and suppose we are considering contracts other than the boundary $F = p = R$, which we have shown in the proof of the first part of Proposition 2 implies a profitable deviation. Hence, here we can assume we are looking for a contract with $F < p, R > p$, L binding or nonbinding. We proceed analogously to the proof of Proposition 1 (case 1) by combining constraints C_1 and C_2 and replacing C_2 by the combined constraint C_2', just like in the proof of Proposition 1, and additionally augment this system by the Euler equation corresponding to the ex post choice of b_2^{η}:

$$C_2^{\eta} : E_2^{\eta} = u'(c_1(R, b_1, b_2))(1 - \rho R_{-1} - F)1 - R - \eta\beta(1-p)u'(c_3(b_2)).$$

We then form the following Lagrangian:

$$\begin{aligned}
\mathcal{L} &= U^\eta(F,R,b_1) + \\
C_3 &: \mu(\pi_1(F,b_1) + (1-p)\pi_2(R,b_2^\eta)) + \\
C_1 &: \gamma\left(u'(c_1(F,b_1))(1-\rho R_{-1}-F) - \eta\beta(1-p)u'(c_2(R,b_1,b_2))\right) + \\
C_2' &: \xi\left(u'(c_1(R,b_1,b_2))(1-\rho R_{-1}-F) - \frac{\eta\beta(1-p)^2u'(c_3(b_2))}{1-R}\right) + \\
C_2^\eta &: \hat{\xi}\left(u'(c_2(R,b_1,b_2^\eta))(1-R) - \eta\beta(1-p)u'(c_3(b_2^\eta))\right),
\end{aligned} \tag{69}$$

Note that the Lagrangian is different than the one in the proof of Proposition 1 as it additionally includes the ex post consumer's Euler condition for period two, i.e, the constraint C_2^η, since involves an additional choice variable b_2^η.

Case 1: $L > 0$ is nonbinding in positive credit economy, i.e., $U_{b_1} = 0$.

The Karush-Kuhn-Tucker (KKT) *comprise stationarity*:

$$F \;:\; U_F + \mu\pi_{1F} - \gamma\left(u_1''(1-\rho R_{-1}-F)b_1)u_1'\right) - \xi(u_1''(1-\rho R_{-1}-F)b_1 + u_1')(1-\rho R_{-1}-F) = 0 \tag{70}$$

$$R \;:\; U_R + \mu(1-p)\pi_{2R} + \gamma\left(\beta(1-p)u_2''b_2\right) - \xi\left(\frac{\eta\beta(1-p)^2u_3''}{(1-R)^2}\right) = 0 \tag{71}$$

$$b_1 \;:\; U_{b_1} + \gamma\left(u_1''(1-\rho R_{-1}-F)^2 + \beta(1-p)u_2''\right) + \xi\left(u_1''(1-\rho R_{-1}-F)^2\right) + \hat{\xi}\left(-u_2''(1-R)\right) = -\mu\pi_{1b_1} \tag{72}$$

$$b_2 \;:\; -\gamma\beta(1-p)u_2''(1-R) + \xi\left(\frac{\eta\beta((1-p))^2u_3''}{1-R}\right) = 0, \tag{73}$$

$$\begin{aligned}
b_2^\eta \;:\; & -\gamma\left(\beta(1-p)u_2''(1-R)\right) + \\
& \hat{\xi}\left(u_2^{\eta\prime\prime}(1-R)^2 + \eta\beta(1-p)u_3^{\eta\prime\prime}\right) = -\mu(1-p)\pi_{2b_2^\eta},
\end{aligned} \tag{74}$$

dual feasibility: $\mu \geq 0,\ \gamma \geq 0,$ *complementarity slackness*:

$$\begin{aligned}
\mu\left(\pi_1(F,b_1) + (1-p)\pi_2(R,b_2^\eta)\right) &= 0 \\
\gamma\left(u'(c_1(F,b_1))(1-\rho R_{-1}-F) - \beta(1-p)u'(c_2(R,b_1,b_2))\right) &= 0,
\end{aligned}$$

and *primal feasibility*: constraints C_3, C_1, C_2' and C_2^η. If CQ holds globally, these conditions are sufficient conditions for an extremum and allow to identify all contracts on the interior of Θ that are suspect to be a global maximum.

In this case we have $U_{b_1} = 0$, since first period Euler equation is assumed to hold with equality. The first set of stationarity conditions are similar as in the proof of Propositions 1, and we proceed analogously to obtain $\gamma = \xi = 0$. Using $\gamma = \xi = 0$, and the fact that $\mu > 0$, similarly gives

$$\frac{U_F^\eta}{U_R^\eta} = \frac{\pi_{1F}}{(1-p)\pi_{2R}}. \tag{75}$$

Using the envelope condition derived for hyperbolic preferences, we obtain

$$\frac{U_F^\eta}{U_R^\eta} = \frac{u_1'\pi_{1F}}{\beta\eta(1-p)u_2'\pi_{2R}}\frac{b_2^\eta}{b_2}.$$

Combing the above conditions, we obtain *lender optimality condition* of the form:

$$\begin{aligned}\frac{u_1'\pi_{1F}}{\beta\eta(1-p)u_2'\pi_{2R}}\frac{b_2^\eta}{b_2} &= \frac{\pi_{1F}}{(1-p)\pi_{2R}}\\ u_1' &= \beta\eta u_2'\frac{b_2}{b_2^\eta}.\end{aligned}$$

As before, the first period Euler equation, i.e., constraint C_1, requires

$$u_1'(1-\rho R_{-1}-F)=\eta\beta(1-p)u_2'.$$

Dividing side-by-side, the two equation imply

$$(1-\rho R_{-1}-F)=(1-p)\frac{b_2^\eta}{b_2}.$$

As far as the proof of Proposition 2 goes, it is straightforward to see that under the stated conditions $\frac{b_2^\eta}{b_2}\to\infty$, which for sufficiently high value implies $F<0$ (in our case corner solution). For proof of Proposition 3, note that, if $R_{-1}>0$, we must have a lower F to satisfy the two equation. The definition of $B(R_{-1}$ ensures that we are comparing allocations such that for the same contract consumption and profits are exactly identical, and hence we can speak about relative magnitudes of F depending on the value of R_{-1}.

We conclude by showing that CQ holds globally. To that end, recall the gradients from (48-50) Part of the argument that is analogous to the one used in the proof of Proposition 1 (see $^\$$ in proof of Proposition 1). To that end, we note that the gradient vectors can be signed as follows:

$$\nabla E_1=\begin{bmatrix}-\\-\\-\\+\\0\end{bmatrix},\nabla E_2'=\begin{bmatrix}-\\+\\-\\-\\0\end{bmatrix},\nabla\tilde{\Pi}=\begin{bmatrix}+\\+\\-^*\\0\\+^*\end{bmatrix},\nabla E_2^\eta=\begin{bmatrix}\frac{\partial E_2^\eta}{\partial F}\\\frac{\partial E_2^\eta}{\partial R}\\\frac{\partial E_2^\eta}{\partial b_1}\\\frac{\partial E_2^\eta}{\partial b_2}\\\frac{\partial E_2^\eta}{\partial b_2^\eta}\end{bmatrix}=\begin{bmatrix}0\\-u_2''b_2^\eta(1-\hat{R})-u_2'\\-u_2''(1-\hat{R})\\0\\u_2''(1-\hat{R})^2+\eta\beta(1-p)u_3''\end{bmatrix}=\begin{bmatrix}0\\-\\+\\0\\-\end{bmatrix}$$

where, as before, $+$ $(-)$ denotes strictly positive (negative) number and * indicates a weak sign.

The first three gradient vectors are simply extended versions of (48)-(50), and we omit the equation. In this case we additionally use LMC to sign the expression in the first entry. Note that the first entry correspond to the derivative of the first period Euler equation and the continuation indirect utility from second period on is concave as a function of b_1.

Consider first $F=p=\hat{R}$, which is only relevant for Proposition 3. We have already shown in the first part of proof of Proposition 2 that this contract cannot be optimal. In such a case the profit function has zeros where * is. It is clear that ∇E_1, $\nabla E_2'$, ∇E_2^η are linearly independent, since ∇E_2^η has a zero first coordinate and ∇E_1, $\nabla E_2'$ have already been shown to be linearly indepdendent in the proof of Proposition 1. It is now straightforward to see that a modified vector $z^T=(0,-1,0,-(1-R)^{-1},a)$, for some $a>0$, meets all the criteria of MFCQ by the same argument as in the proof of Proposition 1 and the fact that the last coordinate of ∇E_1, $\nabla E_2'$, $\nabla\tilde{\Pi}$ are all zeros and we can freely use a to ensure $\nabla E_2^{\eta T}z=0$.

Consider now $F<p,R>p$. We use linear independence CQ. It is clear that gradients ∇E_1, $\nabla E_2'$, $\nabla\tilde{\Pi}$ are linearly independent. This follows from the fact that we have already shown ∇E_1, $\nabla E_2'$ are linearly independent in the proof of Lemma 7, and these vectors have a zero as their last entry while $\nabla\tilde{\Pi}$ has a strictly positive entry given $(1-p)(\hat{R}-p)>0$. What remains to be shown is that we cannot find scalars z_1,z_2,z_3, not all zero, such that $z_1\nabla E_1+z_2\nabla E_2'+z_3\nabla\tilde{\Pi}=\nabla E_2^\eta$. The last row of this system immediately implies $z_3>0$, since it requires $z_3\times(+)=(-)$. Signs in the fourth row of this equation imply z_1 and z_2

must be of the same sign and nonzero ($z_1 \times (-) + z_2 \times (+) = 0$). But then the third row implies $z_1 < 0$, $z_2 < 0$ ($z_1 \times (-) + z_2 \times (-) + z_3 \times (-) = (+)$). This is a contradiction because the first row yields a strictly positive number but must yield a zero ($z_1 \times (-) + z_2 \times (-) + z_3 \times (+) = (0)$).

Case 2: Assume L binds, and hence $U_{b_1} > 0$. We consider an analogous Lagrangian except that we do not combine constraints C_1 and C_2. The Karush-Kuhn-Tucker (KKT) stationarity conditions – after removing C_1 from active constraints – comprise:

$$\begin{aligned}
F &: \quad U_F^\eta + \mu\pi_{1F} = 0 && (76)\\
R &: \quad U_R^\eta + \mu(1-p)\pi_{2R} + \xi\left(-u_2'' b_2^+ (1-R) - u_2'\right) = 0 && (77)\\
b_1 &: \quad U_{b_1}^\eta + \xi\left(-u_2''(1-R)\right) = -\mu\pi_{1b_1} && (78)\\
b_2 &: \quad \xi\left(u_2''(1-R)^2 + \ \beta(1-p)u_3''\right) = 0, && (79)\\
b_2^\eta &: \quad \xi'\left(u_2^{\eta\prime\prime}(1-R)^2 + \eta\beta(1-p)u_3^{\eta\prime\prime}\right) = -\mu(1-p)\pi_{2b_2^\eta}, && (80)
\end{aligned}$$

$\xi = 0$ by equation (79). Accordingly, since $\mu > 0$, the system implies

$$\frac{U_F^\eta}{U_R^\eta} = \frac{\pi_{1F}}{(1-p)\pi_{2R}}.$$

which after substituting out U_F^η / U_R^η from envelope condition in (68) reduces to

$$u_1' = \beta\eta u_2' \frac{b_2}{b_2^\eta}. \tag{81}$$

Using equation (76), and the envelope condition with respect to F, we obtain $u_1 = \mu$. Combining with (78), after plugging in for U_{b_1} that by envelope theorem is $U_{b_1} = u_1'(1 - R_{-1} - F) - \eta\beta(1-p)u_2'$, we obtain

$$u_1'(1 - R_{-1} - F) - \eta\beta(1-p)u_2' = -u_1'(F - p).$$

Dividing both sides by u_1' and using (81), we obtain

$$(1 - R_{-1} - F) - (1-p)\frac{b_2^\eta}{b_2} = -(F - p),$$

which after simplifications gives

$$1 - p - R_{-1} = (1-p)\frac{b_2^\eta}{b_2}.$$

This is a contraction because for $\eta < 1$ we have $\frac{b_2^\eta}{b_2} > 1$.

We next verify CQ to ensure there are no other extrema that do not satisfy KKT. We use linear independence CQ. Recall that the active constraints are:

$$\begin{aligned}
E_2 &= u'(c_2(R, b_1, b_2))(1-R) - \eta\beta(1-p)u'(c_3(b_2))\\
E_2^\eta &= u'(c_2(R, b_1, b_2^\eta))(1-R) - \eta\beta(1-p)u'(c_3(b_2^\eta)),\\
\tilde{\Pi} &= \pi_1(F, b_1) + (1-p)\pi_2(R, b_2^\eta),
\end{aligned}$$

Using local monotonicity to sign the derivative of the Euler equation as negative, and using 50, we have

$$\nabla E_2 = \begin{bmatrix} \frac{\partial E_2}{\partial F} \\ \frac{\partial E_2}{\partial R} \\ \frac{\partial E_2}{\partial b_1} \\ \frac{\partial E_2}{\partial b_2} \end{bmatrix} = \begin{bmatrix} 0 \\ u_2''(1-R)^2 - u_2' \\ -u_2' \\ \beta(1-p)u_3'' \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ - \\ - \\ - \\ 0 \end{bmatrix}, \nabla E_2^\eta = \begin{bmatrix} 0 \\ u_2''(1-R)^2 - u_2' \\ -u_2' \\ 0 \\ \beta(1-p)u_3'' \end{bmatrix} = \begin{bmatrix} 0 \\ - \\ - \\ 0 \\ - \end{bmatrix}, \nabla \Pi^\eta = \begin{bmatrix} + \\ + \\ -^* \\ 0 \\ +^* \end{bmatrix}$$

We note that the profit function $\tilde{\Pi}$ does not depend on b_2 while E_2 strictly negatively depends on it (fourth coordinate is zero), which implies these two vectors must be linearly independent. Suppose now that it is possible to find scalars z_1 and z_2 not all zero such that $z_1 \nabla E_2 + z_2 \nabla \tilde{\Pi} = \nabla E_2^\eta$. Note that neither E_2^η nor E_2 depend on F (first coordinate is zero), while $\tilde{\Pi}$ positively depends on it, and hence we must have $z_2 = 0$. But, if so, the fact that E_2 is independent of b_2^η and E_2' depends on b_2^η (last coordinates are zero) requires $z_1 = 0$, a contradiction.

Refinancing in the limit:

We now show that refinancing take place in the limit under the stated conditions. We have shown that eventually $F = 0$ (case 1). But, if so, the lender breaks even iff $(1-p)(\hat{R}-p)\frac{b_2^\eta}{b_1} > p$, which implies $\hat{R} = p(1 + (\frac{b_2^\eta}{b_1} - p\frac{b_2^\eta}{b_1})^{-1}$. The refinancing threshold is $p/(1-\rho)$, and hence the consumer refinances when $\rho < (1 + \frac{b_2^\eta}{b_1}(1-p))^{-1}$. ■