(Not intended for publication) Online Appendix for "Understanding Growth Through Automation: The Neoclassical Perspective"*

Lukasz A. Drozd
Federal Reserve Bank of Philadelphia

Mathieu Taschereau-Dumouchel
Cornell University

Marina M. Tavares
International Monetary Fund

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Abstract

This document contains the supplementary (online) appendix for the paper "Understanding Growth Through Automation: The Neoclassical Perspective."

Key words: Automation, labor share, Uzawa's theorem, Cobb-Douglas production function, capital-augmenting technological progress, balanced growth

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*The views expressed herein are solely those of the authors and do not necessarily reflect those of the International Monetary Fund, the Federal Reserve Bank of Philadelphia or the Federal Reserve System. Drozd (corresponding author): lukasz.drozd@phil.frb.org, Research Department, Federal Reserve Bank of Philadelphia, Ten Independence Mall, Philadelphia, PA 19106-1574; Taschereau-Dumouchel: Department of Economics, Cornell University, mt763@cornell.edu. Mendes Tavares: Research Department, International Monetary Fund, MMendesTavares@imf.org. The paper has been inspired by Guido Menzio's Society for Economic Dynamics 2021 plenary address. All errors are ours.

Appendix A: Normalization of labor requirement

Here we sketch the argument of how our assumption that the labor requirement is fixed across tasks can be thought of as a normalization in a more general setup with labor and capital requirements that differ by task but are mutually independent.

Suppose there is a separate capital and labor requirement function $k(q)$ and $l(q)$. Define $\hat{k}(q) := k(q)/l(q)$. Let q be ordered so that $\hat{k}(q)$ is increasing as assumed in text. Suppose these functions are measurable and the labor requirement is independent of (relative) capital requirement; that is, knowing $\hat{k}$ gives no information about l, implying for any q we have $\mathbb{E}\left\{l(q)\,|\hat{k}(q), q \in \mathcal{I}\right\} = \mathbb{E}\{l(q)\}$, where $\mathcal{I} = [a,b] \subset \mathcal{Q}$ is any bounded interval (we use expectation operator under a probability measure p induced on that interval; a $\sigma-$finite measure generates a conditional probability distribution on a bounded interval).

This implies that there exists a constant $C > 0$ such that for any bounded interval $\mathcal{I} = [a,b] \subset \mathcal{Q}$ we have

$$\int_{\mathcal{I}} l(q)\, d\mu = \int_{\mathcal{I}} C d\mu, \tag{1}$$

by assumption, since l is i.i.d. with respect to q (if this is not the case it would be possible to infer l from $\hat{k}$—that is, from q since $\hat{k}(q)$ is increasing—and we assume here that $\hat{k}(q)$ is strictly increasing on at least part of the domain).

Let us now normalize units of capital requirement and labor requirement by some positive constant $C > 0$; that is, abusing notation a bit, (re)define $k(q) := k(q)/C$ and $l(q) := l(q)/C$. Note that this only a change of units in which inputs are measured to ensure, by (1), that $\int_{\mathcal{I}} l(q)\, d\mu = \int_{\mathcal{I}} 1 d\mu$, as in the paper. On any bounded interval $\mathcal{I} = [a,b]$ we have

$$\int_{\mathcal{I}} k(q)\, d\mu = \int_{\mathcal{I}} \hat{k}(q)\, l(q)\, d\mu = \int_{\mathcal{I}} \hat{k}(q)\, d\mu \int_{\mathcal{I}} l(q)\, d\mu = \int_{\mathcal{I}} \hat{k}(q)\, d\mu,$$

where the first equality follows by definition of $\hat{k}$, the second equality follows from independence, and the last inequality follows from (1) by normalization. We have now obtained the result by showing that inputs are the same on the redefined and normalized space as on the original space.[1] We omit the details of extending this result to $\mathscr{B}(\mathcal{Q})$, which is standard but

[1] It must also be the case that the information we dropped is irrelevant for the firm, which we assume is the case. As a counterexample, suppose the firm—for whatever reason—chooses to do tasks with capital iff $l \geq 5$. In

cumbersome.

Appendix B: Omitted proofs of Lemmas 1, 2 and parts of 3

Proof of Lemma 1 **Part I:** We first show that technological constraints given by equation 4 in the paper are satisfied for finite inputs when a cutoff rule is used; in particular, we need to show that (i) the constant function $f(q) = 1$ is μ-integrable, or equivalently $g(q)$ is Lebesgue integrable on interval $[q^*, \infty)$, and that (ii) $k(q)$ is μ-integrable, or equivalently $k(q)\, g(q)$ is Lebesgue integrable on interval $[0, q^*]$, where q^* satisfies the requirement of the lemma. We prove it in two steps.

Step 1: Assume $q^* < \infty$ and $k(q^*) > 0$. To establish property (i) above, define $\mathcal{S} = [q^*, \infty)$. By contradiction, suppose $g(q)$ is not Lebesgue integrable on $\mathcal{S}$ (i.e., $\int_{\mathcal{S}} d\mu = \int_{\mathcal{S}} g(q)\, dv = +\infty$, since measurability is assumed and g is a non-negative function). By Assumption 1 in the paper, we know there is a measurable partition of $\mathcal{S}$ comprising two disjoint subsets $\mathcal{S}_l = \mathcal{Q}_l \cap \mathcal{S}, \mathcal{S}_k = \mathcal{Q}_l \cap \mathcal{S}$ such that $\int_{\mathcal{S}_l} d\mu < \infty$ and $\int_{\mathcal{S}_k} k(q)\, d\mu < \infty$, where $\{\mathcal{Q}_l, \mathcal{Q}_k\}$ is the partition implied by Assumption 1. Since $k(q)$ is an increasing function, we know

$$\infty > \int\limits_{\mathcal{S}_k} k(q)\, d\mu \geq \int\limits_{\mathcal{S}_k} k(q^*)\, d\mu = k(q^*) \int\limits_{\mathcal{S}_k} d\mu,$$

which gives a contradiction by the following chain of evaluations:

$$\infty > \int\limits_{\mathcal{S}_l} 1 d\mu + \int\limits_{\mathcal{S}_k} k(q)\, d\mu \geq \int\limits_{\mathcal{S}_l} 1 d\mu + k(q^*) \int\limits_{\mathcal{S}_k} d\mu = (1 + k(q^*)) \int\limits_{\mathcal{S}} 1 d\mu = +\infty. \qquad (2)$$

To establish property (ii), we note that $0 \leq k(q) \leq \frac{w}{r} < \infty$ for all $q \leq q^*$, implying $\frac{r}{w} k(q) < 1$. This follows by the definition of cutoff q^* in the statement of lemma. Ac-

that case the ratio k/l would not be sufficient. This is not the case in our model because the firm only maximizes profits.

cordingly, $\int_0^{q^*} k(q)\, d\mu < \infty$ by the following chain of evaluations:

$$\begin{aligned} \infty > \int\limits_{\mathcal{Q}_l} 1 d\mu + \int\limits_{\mathcal{Q}_k} k(q)\, d\mu &\geq \int\limits_{\mathcal{Q}_l} \frac{r}{w} k(q)\, d\mu + \int\limits_{\mathcal{Q}_k} k(q)\, d\mu \\ &= \left(\frac{r}{w} + 1\right) \int\limits_{\mathcal{Q}} k(q)\, d\mu \geq \left(\frac{r}{w} + 1\right) \int_0^{q^*} k(q)\, d\mu. \end{aligned}$$

(For Step 2 below, note that the proof of property (i) actually does not depend on q^* in the statement of the lemma, and the proof of property (ii) does not depend on $k(q^*) > 0$.)

Step 2: This step covers degenerate cases: a) $k(q^*) = 0$ ($0 \leq q^* < \infty$) or b) $q^* = +\infty$ (note: a and b is impossible by Assumption 1, since by that assumption $k(q)$ must be strictly positive for a sufficiently large q).

Case a: By definition of the cutoff in the statement of the lemma and the fact that $k(q)$ is an increasing function, we have $k(q) \geq \frac{w}{r} > 0$ for all $q > q^*$, and $k(q) = 0$ for all $q \leq q^*$ (the strictly inequality follows here from the hypothesis that $k(q^*) = 0$). Accordingly, we have established now property (i), since $\int_0^{q^*} k(q)\, d\mu = \int_0^{q^*} 0 d\mu = 0$. Recall that, as noted at the end of Step 1 above, the argument used in Step 1 above (proof of property ii) does not require $k(q^*) > 0$ as assumed in Step 1, and so property (ii) has already been proven there.

Case b: Note that $q^* = \infty$ implies $k(q) \leq \frac{w}{r}$ for all $q \in \mathcal{Q}$ by the cutoff rule stated in the lemma. Accordingly, by Assumption 1 in the paper, and the fact that $\frac{r}{w} k(q) \leq 1$ for all $q \in \mathcal{Q}$, property (i) follows from the evaluation:

$$\infty > \int\limits_{\mathcal{Q}_l} 1 d\mu + \int\limits_{\mathcal{Q}_k} k(q)\, d\mu \geq \frac{r}{w} \int\limits_{\mathcal{Q}_l} k(q)\, d\mu + \int\limits_{\mathcal{Q}_k} k(q)\, d\mu = \left(\frac{r}{w} + 1\right) \int\limits_{\mathcal{Q}} k(q)\, d\mu.$$

To see that $\lim_{q^* \to \infty} \int_{q^*}^{\infty} 1\mu = 0$, we apply the argument used in Step 1 (proof of property i) to show that $\int_{q^{**}}^{\infty} 1\mu$ exists (is finite) for sufficiently large q^{**} such that $k(q^*) > 0$ (the existence of such a sufficiently large and finite q^{**} is ensured by the fact that $k(q)$ is strictly positive on at least part of the domain by Assumption 1 and, as noted, Step 1 (proof property i) did not actually rely on the assumption that q^* corresponds to the cutoff as defined in the statement of lemma). Since for any Lebesgue integral we have $\lim_{q^* \to \infty} \int_{q^*}^{\infty} 1\mu = 0$, we have now shown that both constraints in equation 4 in the paper are well defined when the cutoff rule is used.[2]

Part II: This part establishes that the proposed cutoff rule satisfies cost minimization. By contradiction, suppose that there exists a task partition $\mathcal{Q}_k = \mathcal{E}$ of a positive measure under μ that solves the minimization problem and that is different from that implied by the cutoff rule in the lemma (on a measurable set with a positive measure). If so, reassigning production of tasks in $\mathcal{A} = \mathcal{E} \cap \{q : q > q^*\}$ from capital to labor must reduce the cost because $rk(q) > w$ on that set by definition of the cutoff rule—since we are minimizing $rK + wL$—and analogously on set $\mathcal{A}^c$ on which we switch from using labor to capital. At least one of these sets must be of positive measure, contradicting cost minimization and establishing the result. Q.E.D.

Proof of Lemma 2 Consider the definition of the production function (equation 5 in the paper) with equality:

$$Y(K, L) := \sup \left\{ Y : \exists_{q^* \in \mathcal{Q}} \text{ s.t. } K = Y \int_0^{q^*} k(q)\, d\mu, L = Y \int_{q^*}^{\infty} 1 d\mu \right\}. \quad (3)$$

We split the proof to two steps: Step 1 shows the solution (Y,q^*) to the above equations exists. Step 2 shows the solution from Step 1 attains the supremum under the formulation stated in the paper (equation 5 in the paper).

Step 1: Note that the constraint in (3) implies that q^* satisfies

$$\frac{L}{K} = \frac{\int_{q^*}^{\infty} 1 d\mu}{\int_0^{q^*} k(q)\, d\mu}. \quad (4)$$

The integral in the numerator is finite whenever the integral in the denominator is nonzero. We have established this property in the proof of Lemma 1 (see Part I, Step 1). The key here is that when the denominator (or $K > 0$) is positive, then $k(q^*) > 0$, which in turn implies the existence (finiteness) of the integral in the numerator by the arguments used in the proof of Lemma 1 (see Part I, Step 1, proof of property i). Next, note the following basic properties of

[2] The proof follows the fact that the tail sum of any convergent series converges to zero, that is, if $\sum_{i=1}^{\infty} a_i$ converges, then $t_n = \sum_{i=n}^{\infty} a_i \to_{n\to\infty} 0$, which itself is a corollary from Cauchy's criterion of convergence for series. Specifically, define $a_i = \int_{q^*+i-1}^{q^*+i} 1 d\mu$, note that $\sum_{i=1}^{\infty} a_i = \int_{q^*}^{\infty} 1\mu < \infty$ by Theorem 5.24 from Wheeden and Zygmund (1977) and the hypothesis, and now apply the result for series.

the expression on the right-hand side of equation (4): i) the numerator can be made arbitrarily small as $q^* \to 0$, and since the numerator is increasing in q^*, the expression goes to ∞ as $q^* \to 0$; ii) the numerator goes to 0 when $q^* \to \infty$, and since the denominator is positive and increasing in q^*, the expression goes to 0 as $q^* \to \infty$ (the proof of this simple fact can be found in footnote 2); finally, iii) note that the expression is continuous with respect to q^* and strictly decreasing, and by all these properties it is bijective on $\mathbb{R}_+$.[3] Accordingly, there exists a unique $0 < q^* < +\infty$ that satisfies the two constraints (for any finite $L/K > 0$). Furthermore, the supremum is attained within this set. (Without a loss we can restrict attention to a compact domain of $\left(\hat{Y}, q^*\right)$ while maximizing a continuous function $f\left(\hat{Y}\right) = \hat{Y}$ on the set defined by (3). Accordingly, Weierstrass extreme value theorem ensures the existence of maximum.)

Step 2: We now turn to the question of whether this solution attains the supremum under the original definition of the production function given by equation 5 in the paper. For now assume $K > 0$. (We cover $K = 0$ at the very end.) Suppose, by the way of contradiction that there exists $\hat{Y}' > \hat{Y}$, $q^{*\prime} > 0$ such that $K > \hat{Y}' \int_0^{q^{*\prime}} k(q)\, g(q)\, dv, L \geq \hat{Y}' \int_{q^{*\prime}}^{\infty} g(v)\, dv$ (the case $K = \hat{Y}' \int_0^{q^{*\prime}} k(q)\, g(q)\, dv$, $L > \hat{Y}' \int_{q^{*\prime}}^{\infty} g(v)\, dv$ will follow by analogy and it is omitted). If so, the supremum of the original problem must exceed the one implied by 3, which, as we show next, leads to a contradiction. Note that the integrals exist at $q^{*\prime}$ by the hypothesis (the stated inequalities guarantee these integrals are finite). By the continuity of Lebesgue integrals (see footnote 3), we can pick $\Delta q^{*\prime} > 0$ such that $K > \hat{Y} \int_0^{q^{*\prime}+\Delta q^{*\prime}} k(q)\, g(q)\, dv$, which implies that there exists $\Delta\hat{Y} > 0$ such that $K = \left(\hat{Y}' + \Delta\hat{Y}'\right) \int_0^{q^{*\prime}+\Delta q^{*\prime}} k(q)\, g(q)\, dv$ (by continuity of the expression on the right). We must ensure that the integral in the last expression exists (is finite). Let $\bar{k} := \sup_{[q^{*\prime}, q^{*\prime}+\Delta q^{*\prime}] \subset \mathcal{Q}} k(q)$, which, note, must be a finite number. (If this was not the case, we would have had $k(q^{**}) = +\infty$ for any $q^{**} > q^{*\prime} + \Delta q^{*\prime}$—simply because $k(q)$ is increasing and it is defined everywhere on $\mathcal{Q}$.) The following chain of evaluations now shows that the integral in question exists as long as $\int_{q^{*\prime}}^{\infty} g(v)\, dv$ exists, which is the case by the hypothesis:

$$\infty > \bar{k} \int_{q^{*\prime}}^{\infty} g(q)\, dq > \bar{k} \int_{q^{*\prime}}^{q^{*\prime}+\Delta q^{*\prime}} g(q)\, dq > \int_{q^{*\prime}}^{q^{*\prime}+\Delta q^{*\prime}} k(q)\, g(q)\, dq.$$

[3] Lacking a textbook reference, we prove it in the Online Appendix E.

Returning to the main argument, the fact that g has full support implies $L > \hat{Y}' \int_{q^{*\prime}+\Delta q^{*\prime}}^{\infty} 1 g(v)\, dv$ by continuity of Lebesgue integrals.[4] But, if so, there exists $\hat{Y}'' = \hat{Y}' + \Delta\hat{Y}''$, for some $0 < \Delta\hat{Y}'' < \Delta\hat{Y}'$, such that $K \geq \hat{Y}'' \int_0^{q^{*\prime}+\Delta q^{*\prime}} k(q)\, g(q)\, dv$ and we maintain $L = \hat{Y}'' \int_{q^{*\prime}+\Delta q^{*\prime}}^{\infty} g(v)\, dv$, which is a contradiction of the fact that $\hat{Y}' > \hat{Y}$. $\hat{Y}' = \infty$ is not feasible because k is strictly positive on at least part of the domain (see Assumption 1 in the paper). The remaining case is easy to eliminate by instead considering "$-\Delta q^{*\prime}$" and we omit the details. If $K = 0$, note, there is not much to prove because $q^* = 0$. Q.E.D.

Proof of Lemma 3 (omitted parts from the paper) Part I shows existence and Part II derives the formulas and the proof is in the paper. **Part I:** The proof builds on the proof of Lemma 2. We have established in that lemma that the production function can be obtained from (3) and that a unique q^* exists that satisfies (4). By the second constraint then, we know that $Y(K, L), q^*$ satisfy

$$L = Y(K, L)\, \mu([q^*, \infty)), \tag{5}$$

which gives

$$q^*(Y, L) = S^{-1}\left(\frac{L}{Y}\right), \tag{6}$$

where $S(q) := \mu([q, \infty))$ is the survival function. The survival function under the assumptions made in the paper, by previous lemmas, is well-defined, positively-valued, continuous, strictly decreasing (because g has full support), and hence invertible and differentiable almost everywhere with a strictly negative derivative.[5] Accordingly, $S^{-1}\left(\frac{L}{Y}\right)$ exists and is differentiable a.e., since for functions of a single variable we have $[f^{-1}]'(x) = \frac{1}{f'(f^{-1}(x))}$, which is well defined as long as f' is nonzero (which it is). This implies that the derivative of $q^*(Y, L)$ in (6) is well defined (a.e.). The production function $Y(K, L)$ can be recovered from capital usage equation of Lemma 2, which gives the identity:

$$f(Y(K, L), K) := Y(K, L) \int_0^{q^*(Y(K,L),L)} k(q)\, d\mu - K \equiv 0.$$

[4] As in footnote 3.

[5] See Theorem 7.21 (p. 111) in Wheeden and Zygmund (1977).

The implicit function theorem ensures that at the points of differentiability of $q^* (Y, L)$ the partial derivative $Y_K (K, L)$ is well defined as long as the partial derivative $f_Y (Y, K) := \frac{\partial f(Y,K)}{\partial Y}$ is non-vanishing (nonzero) and both f_Y, f_K are well defined—which is readily implied by the above functional form. The existence of the partial derivative with respect to L can be shown analogously and we omit it. **Part II:** In the appendix of the paper.

Appendix C: Growth properties of domain-truncated CD technology

Here we show how to obtain approximately balanced growth from the domain-truncated Cobb-Douglas task technology (TCD) of the form:

$$T_{q_0}^{TCD} = \left(\mathcal{Q} = [q_0, \infty), k(q) = Z^{-1} q^{\frac{1}{\alpha}} \frac{1-\alpha}{\alpha}, g(q) = A^{-1} q^{-2} \right), \tag{7}$$

where $q_0 > 0$. (For convenience, we modify domain $\mathcal{Q}$ instead of adding q_0 to effectively also shift the task domain.)

In this case the density function can be normalized by a constant to yield the standard Pareto probability density, implying that the implied measure μ is finite, and hence $T_{q_0}^{TCD}$ has probabilistic representation. We will show that this technology gives rise to approximately balanced growth and its predictions can be made statistically indistinguishable from the balanced growth path of the CD economy by picking sufficiently small q_0 given a finite sample of data.

To derive the aggregate production function implied by TCD technology, we follow the steps in Example 1 of the paper. If $q^* > q_0$, we obtain the following equation for the representative isoquant:

$$\frac{K}{Y}\left(\frac{L}{Y}\right) := (AZ)^{-1} \left(\left(A\frac{L}{Y} \right)^{1-\frac{1}{\alpha}} - q_0^{\frac{1}{\alpha}-1} \right). \tag{8}$$

As expected, $q_0 \to 0$ implies the production function is CD. However, the constraint $q^* \geq q_0$ may be binding, and in that case the above equation does not apply because Lemma 2 does not apply. Accordingly, we use the original definition of the production function and obtain $Y = ALq_0$, $K = 0$ when $\left(A\frac{L}{Y}\right)^{1-\frac{1}{\alpha}} \leq q_0^{\frac{1}{\alpha}-1}$, or equivalently $\frac{L}{Y} \geq (Aq_0)^{-1}$, which implies $q^* = q_0$ is binding. Figure 1 illustrates the obtained isoquant.

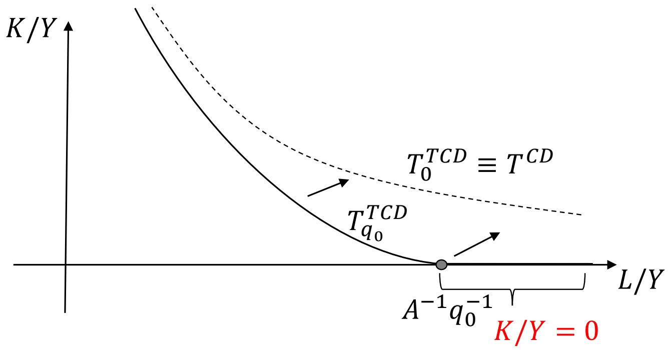

Figure 1: Representative isoquant of the production function implied by the $T_{q_0}^{TCD}$ technology.

Notes: The figure plots the representative isoquant of the production function implied by task technology $T_{q_0}^{TCD}$ (in text). The arrows indicate uniform convergence to CD isoquant as $q_0 \to 0$. On the flat portion, capital is not used in production and output is produced exclusively using labor.

The key property of this isoquant is that it *uniformly* converges to the CD isoquant ($q_0 = 0$) both when $q_0 \to 0$ and $AZ \to \infty$. This follows from the fact that we can bound the difference between the two isoquants by the expression:

$$\sup_{L/Y \geq 0} |\frac{K}{Y}\left(\frac{L}{Y}; q_0\right) - \frac{K}{Y}\left(\frac{L}{Y}; 0\right)| < \sup_{L/Y \geq 0} \frac{1}{\alpha^{-1} - 1} \left(AZ\right)^{-1} q_0^{\frac{1}{\alpha}-1},$$

where $\frac{K}{Y}\left(\frac{L}{Y}; q_0\right)$ is the representative isoquant of the truncated CD technology ($q_0 > 0$) and $\frac{K}{Y}\left(\frac{L}{Y}; 0\right)$ is the representative isoquant of the CD technology. This bound follows from the fact that the "gap" between the two isoquants is decreasing with respect to $\frac{L}{Y}$ above $\frac{L}{Y} \geq \left(Aq_0\right)^{-1}$, as shown in the figure, and thus it is bounded by the "gap" at $\frac{L}{Y} = \left(Aq_0\right)^{-1}$, which itself narrows with $q_0 \to \infty$. This property does not imply that the production function converges uniformly, but it does imply that the production function converges uniformly on an arbitrarily bounded range of inputs. As we show next, after dividing each variable by the growth rate of technology, the model implies that the vector field on the phase space for normalized capital and consumption uniformly converges to that of the CD model.

Growth properties of domain-truncated CD task technology

Assume that Z_t and A_t grow at constant and strictly positive rates $\gamma_Z > 0, \gamma_A > 0$, respectively. Assume that q_0 is sufficiently small so that capital is used in equilibrium; that is, the economy stays on the increasing portion of the isoquant in Figure 1. We return to this at the end. To focus on how the growth path relates to the balanced growth path under CD technology, divide all variables except for labor by the balanced growth factor $(A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}}$ of the CD model. For example, after this normalization, K_t becomes $(A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}}\,\bar{K}_t$, C_t becomes $(A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}}\,\bar{C}_t$, and so on and so forth. Since A and Z both grow at strictly positive rates γ_A, γ_Z, $(A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}}$ grows at rate γ can be easily calculated by differentiating this expression with respect to time. The normalized allocation solves the planning problem of the form:

$$\max_{(C_t,K_t)_t} \int_0^{\infty} e^{-(\rho+\gamma)t} u\left(\bar{C}_t\right) \tag{9}$$

subject to

$$\bar{C}_t + \dot{\bar{K}}_t + \gamma \bar{K}_t - \delta \bar{K}_t = \bar{Y}_t, \tag{10}$$

given $\bar{K}_0$, and $\bar{C}_t \geq 0$, $\bar{K}_t \geq 0$, where, by (8), output $\bar{Y}_t$ solves[6]

$$\bar{Y}_t = \bar{Y}_t\left(\bar{K}_t, \bar{L}; q_0\right) := \left(\frac{\bar{K}_t}{\bar{X}_t\left(\bar{Y}_t, \bar{L}; q_0\right)}\right)^{\alpha} \bar{L}^{1-\alpha}, \tag{11}$$

and where

$$\bar{X}_t\left(\bar{Y}_t, \bar{L}_t; q_0\right) := 1 - (A_t Z_t)^{-1} \left(\frac{\bar{L}}{\bar{Y}_t}\right)^{\frac{1}{\alpha}-1} q_0^{\frac{1}{\alpha}-1}. \tag{12}$$

We refer to the above model as the TCD model while referring to the model in text as the CD model (which is the above but with $q_0 = 0$).

The fixed point that defines $\bar{Y}_t$ exists and is unique—as long as q_0 is not too high, which we assume. This follows from plugging (12) into (11) and noting the opposing monotonicity of the left- and right-hand side of the resulting equation with respect to $\bar{Y}$. Second, the above equation implies that the production function defined by (11) converges to the CD production function uniformly on any bounded domain, in particular for $\bar{\bar{K}} \geq \bar{K} \geq \bar{K}_0 > 0$ ($\bar{L}$ fixed).[7] The addition of an upper bound constraint $\bar{\bar{K}}$ is without a loss given that a sufficiently high

[6]See the Online Appendix B for an explicit derivation of the above formula.

[7]After plugging in from (11) to (12), it can be shown that for sufficiently low q_0 and $\bar{K} \geq K_0$, we can always find a unique X that solves the resulting equation. Plugging in that X to (11), we obtain unique value of output. In addition, X converges to 1 with both $q_0 \to 0$ and $A_t Z_t \to \infty$ for any $\bar{Y}_t > 0$ (uniformly after imposing a lower bound on $\bar{Y}_t$).

level of capital is unsustainable by the assumptions that consumption must be nonnegative and depreciation is a fraction of capital stock. The lower bound follows from our focus on a positive growth equilibrium. We return to this at the end of the section.

Equation (10) implies that the growth rate of capital $\gamma_{\bar{K},t}$ is

$$\gamma_{\bar{K},t}\left(\bar{K}_t,\bar{C}_t;q_0\right) := \frac{\dot{\bar{K}}_t}{\bar{K}_t} = \frac{\bar{Y}_t\left(\bar{K}_t,\bar{L};q_0\right) + (\delta-\gamma)\bar{K}_t - \bar{C}_t}{\bar{K}_t}, \tag{13}$$

and hence, by the observations made, it converges uniformly to the growth rate at $q_0 = 0$ on the bounded domain $\bar{\bar{K}} \geq \bar{K} \geq \bar{K}_0$, both with respect to $q_0 \to 0$ and/or $t \to \infty$ (by which we mean $A_t Z_t \to \infty$). As a result, the growth rate of capital converges uniformly to the CD case, implying

$$\sup_{\bar{\bar{K}} \geq \bar{K} \geq \bar{K}_0, \bar{C} \geq 0} |\gamma_{\bar{K},t}\left(\bar{K},\bar{C};q_0\right) - \gamma_{\bar{K},t}\left(\bar{K},\bar{C};0\right)| \to_{q_0 \to 0, t \to \infty} 0.$$

The Euler condition for the planning problem implies that the growth rate of consumption $\gamma_{\bar{C},t}$ is

$$\gamma_{\bar{C},t}\left(\bar{K}_t,\bar{C}_t;q_0\right) := \frac{\dot{\bar{C}}_t}{\bar{C}_t} = \frac{1}{\sigma}\left(MPK_t\left(\bar{K}_t,\bar{L};q_0\right) - \delta - \rho\right), \tag{14}$$

which, after basic manipulations detailed in the Online Appendix D below, can be linked to $MPK_t\left(\bar{K}_t,\bar{L};0\right)$ as follows

$$MPK_t\left(\bar{K}_t,\bar{L};q_0\right) = \left(\bar{X}_t\left(Y_t,\bar{L};q_0\right)^{\alpha-1} MPK_t\left(\bar{K}_t,\bar{L};0\right)^{-1} - \left(A_t Z_t\right)^{-1} q_0^{\frac{1}{\alpha}-1}\right)^{-1}, \tag{15}$$

where

$$MPK_t\left(\bar{K}_t,\bar{L};0\right) = \alpha\left(\frac{\bar{L}}{\bar{Y}\left(\bar{K}_t,\bar{L};0\right)}\right)^{\frac{1}{\alpha}-1}.$$

Accordingly, we similarly obtain uniform convergence of consumption growth rate:

$$\sup_{\bar{K} \geq K_0} |\gamma_{\bar{C},t}\left(\bar{K},\bar{C};q_0\right) - \gamma_{\bar{C},t}\left(\bar{K},\bar{C};0\right)| \to_{q_0 \to 0, A_t Z_t \to \infty} 0.$$

C and K are the two variables that define the phase space of the dynamic system that solves (9). As a result, as shown in Figure 2, the vector field for this system is a perturbed version of

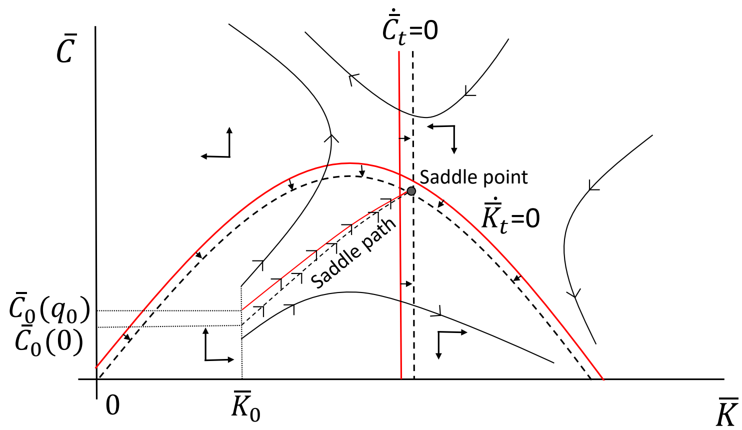

Figure 2: Phase diagram of the growth model with $q_0 > 0$ versus $q_0 = 0$ (dotted line).

Notes: The figure shows a phase diagram implied by the growth model featuring truncated Cobb-Douglas technology ($q_0 > 0$, solid lines) versus exact Cobb-Douglas technology featuring infinite measure ($q_0 = 0$, dotted lines). As shown in text, for $K > K_0$ all objects of the phase diagram exhibit uniform convergence to those associated with exact Cobb-Douglas technology, both with respect to $q_0 \to 0$ as well as time $t \to \infty$ (equivalently $A_t Z_t \to \infty$). Consequently, the optimal time path of capital and consumption along the saddle path is also similar as shown.

the one associated with the CD technology, with that perturbation uniformly vanishing with respect to both $q_0 \to 0$ and $A_t Z_t \to \infty$. Since qualitatively the phase diagram is standard, the solution that satisfies the usual transversality condition and nonnegativity conditions is the saddle path towards the intersection of the loci of points that imply stationary consumption and capital in the long-run. By the continuous dependence on the initial data theorem for differential equation, then, the time paths of each variable approach the CD case, and in the limit converge towards the common saddle point.

Let us now return to the omitted issue of capital being used along the growth path. When $K = 0$, note, the TCD technology implies that $MPK = \left(\frac{1}{\alpha} - 1\right)^{-1} AZ q_0^{\frac{1}{\alpha}-1}$, which together with the Euler equation implies that capital will be accumulated as long as $MPK = \left(\frac{1}{\alpha} - 1\right)^{-1} AZ q_0^{\frac{1}{\alpha}-1} > \rho + \delta + \sigma\gamma_A$, since consumption grows at rate γ_A when capital is not used nor accumulated ($K = 0$), and in that case consumption equals output, i.e., $C = Y = A\bar{L}q_0$. We can ensure this condition holds for q_0 sufficiently low given $A_0 Z_0$ as assumed, and because AZ grows at a strictly positive rate, we can be sure this condition will hold thereafter.

The global transitional dynamics implied by the TCD model is more complicated but it is appealing in its own right. In particular, this model can generate a stylized industrial revolution along the lines of Hansen and Prescott (2002) at low levels of capital and productivity Z. This happens when q_0 is not too low and AZ keeps on growing so that eventually capital starts being used in production (which gives rise to a stylized industrial revolution). The model can also generate a poverty trap when growth in Z comes from learning-by-doing externality associated with using capital as in Romer (1986).

Appendix D. Supplementary derivations for Online Appendix C

Production function for TCD task technology normalized by balanced growth factor: Plugging in $K_t = (A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}} \bar{K}_t$, Y_t= $(A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}} \bar{Y}_t$, $C_t = (A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}} \bar{C}_t$ to the equation for TCD isoquant in text, we obtain

$$\frac{(A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}} \bar{K}_t}{(A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}} \bar{Y}_t} = (A_t Z_t)^{-1} \left(\left(A_t \frac{\bar{L}}{(A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}} \bar{Y}_t} \right)^{1-\frac{1}{\alpha}} - q_0^{\frac{1}{\alpha}-1} \right).$$

Simplifying terms and pulling out the first term in the last bracket, while raising both sides to the power α, we get

$$\left(A_t Z_t \frac{\bar{K}_t}{\bar{Y}_t} \right)^{\alpha} = \left(A_t \frac{\bar{L}}{(A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}} \bar{Y}_t} \right)^{\alpha-1} \left(1 - q_0^{\frac{1}{\alpha}-1} \left(\frac{A_t \bar{L}}{(A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}} \bar{Y}_t} \right)^{\frac{1}{\alpha}-1} \right)^{\alpha},$$

which, given the fact that

$$\frac{A_t}{(A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}}} = \frac{A_t^{1-\frac{1}{1-\alpha}}}{Z_t^{\frac{\alpha}{1-\alpha}}} = \frac{A_t^{-\frac{\alpha}{1-\alpha}}}{Z_t^{\frac{\alpha}{1-\alpha}}} = (A_t Z_t)^{-\frac{\alpha}{1-\alpha}} = (A_t Z_t)^{-\frac{1}{\frac{1}{\alpha}-1}}$$

simplifies to

$$\left(\frac{\bar{K}_t}{\bar{Y}_t} \right)^{\alpha} = \left(\frac{\bar{L}}{\bar{Y}_t} \right)^{\alpha-1} \left(1 - q_0^{\frac{1}{\alpha}-1} (A_t Z_t)^{-1} \left(\frac{\bar{L}}{\bar{Y}_t} \right)^{\frac{1}{\alpha}-1} \right)^{\alpha},$$

$$\left(\frac{\bar{K}_t}{\bar{Y}_t} \right)^{\alpha} \left(\frac{\bar{L}}{\bar{Y}_t} \right)^{1-\alpha} = \left(1 - q_0^{\frac{1}{\alpha}-1} (A_t Z_t)^{-1} \left(\frac{\bar{L}}{\bar{Y}_t} \right)^{\frac{1}{\alpha}-1} \right)^{\alpha},$$

and gives

$$\left(\bar{K}_t\right)^{\alpha}\left(\bar{L}\right)^{1-\alpha}=\bar{Y}_t\left(1-q_0^{\frac{1}{\alpha}-1}\left(A_tZ_t\right)^{-1}\left(\frac{\bar{L}}{\bar{Y}_t}\right)^{\frac{1}{\alpha}-1}\right)^{\alpha}.$$

After some basic manipulation, the last expression yields the fixed point stated in text:

$$\bar{Y}_t\left(\bar{K}_t,\bar{L}_t;q_0\right)=\left(\frac{\bar{K}_t}{\bar{X}_t\left(Y_t,\bar{L};q_0\right)}\right)^{\alpha}\left(\bar{L}\right)^{1-\alpha}, \tag{16}$$

where

$$\bar{X}_t\left(Y_t,\bar{L};q_0\right)=1-q_0^{\frac{1}{\alpha}-1}\left(A_tZ_t\right)^{-1}\left(\frac{\bar{L}}{\bar{Y}_t}\right)^{\frac{1}{\alpha}-1}.$$

Marginal product of capital MPK:

We use the last equation above and raise both sides to power $\frac{1}{\alpha}$ to obtain

$$\left(\bar{K}_t\right)^{1}\left(\bar{L}\right)^{\frac{1}{\alpha}-1}=\bar{Y}_t^{\frac{1}{\alpha}}-q_0^{\frac{1}{\alpha}-1}\left(A_tZ_t\right)^{-1}\left(\frac{\bar{L}}{\bar{Y}_t}\right)^{\frac{1}{\alpha}-1}\bar{Y}_t^{\frac{1}{\alpha}}$$

hence obtain

$$\bar{K}_t=\bar{Y}_t^{\frac{1}{\alpha}}\bar{L}^{\frac{1}{\alpha}-1}-q_0^{\frac{1}{\alpha}-1}\left(A_tZ_t\right)^{-1}\bar{Y}_t,$$

$$\bar{K}_t=\left(A_tZ_t\right)^{-1}\bar{Y}_t\left(A_tZ_t\bar{Y}_t^{\frac{1}{\alpha}-1}\bar{L}^{\frac{1}{\alpha}-1}-q_0^{\frac{1}{\alpha}-1}\right)$$

and

$$A_tZ_t\bar{K}_t=\bar{Y}_t\left(A_tZ_t\left(\frac{\bar{L}}{\bar{Y}_t}\right)^{1-\frac{1}{\alpha}}-q_0^{\frac{1}{\alpha}-1}\right).$$

The above expression defines the production function $\bar{Y}\left(\bar{K}_t,\bar{L};q_0\right)$ implicitly via the expression:

$$A_tZ_t\bar{K}_t-\bar{Y}_t\left(A_tZ_t\left(\frac{\bar{L}}{\bar{Y}\left(\bar{K}_t,\bar{L};q_0\right)}\right)^{1-\frac{1}{\alpha}}-q_0^{\frac{1}{\alpha}-1}\right)\equiv 0$$

where $\bar{Y}\left(\bar{K}_t, \bar{L}; q_0\right)$ is given by (16). We use implicit function theorem and differentiate the above to calculate[8]

$$MPK_t\left(\bar{K}_t, \bar{L}; q_0\right) = \left(\alpha^{-1}\left(\frac{\bar{L}}{\bar{Y}\left(\bar{K}_t, \bar{L}; q_0\right)}\right)^{1-\frac{1}{\alpha}} - \left(A_t Z_t\right)^{-1} q_0^{\frac{1}{\alpha}-1}\right)^{-1}.$$

For $q_0 = 0$, note, we obtain

$$MPK_t\left(\bar{K}_t, \bar{L}; 0\right) = \alpha\left(\frac{\bar{L}}{\bar{Y}\left(\bar{K}_t, \bar{L}; 0\right)}\right)^{\frac{1}{\alpha}-1},$$

which is the expression for MPK for the CD production function given by $\bar{Y}\left(\bar{K}_t, \bar{L}; 0\right)$. Now, by (16), we know that

$$\bar{Y}\left(\bar{K}_t, \bar{L}; q_0\right) = \bar{X}_t\left(Y_t, \bar{L}; q_0\right)^{-\alpha} \bar{Y}\left(\bar{K}_t, \bar{L}; 0\right).$$

Accordingly, we have

$$MPK_t\left(\bar{K}_t, \bar{L}; q_0\right) = \left(\bar{X}_t\left(Y_t, \bar{L}; q_0\right)^{\alpha-1} \alpha^{-1}\left(\frac{\bar{L}}{\bar{Y}\left(\bar{K}_t, \bar{L}; 0\right)}\right)^{1-\frac{1}{\alpha}} - \left(A_t Z_t\right)^{-1} q_0^{\frac{1}{\alpha}-1}\right)^{-1}$$

and hence

$$MPK_t\left(\bar{K}_t, \bar{L}; q_0\right) = \left(\bar{X}_t\left(Y_t, \bar{L}; q_0\right)^{\alpha-1} MPK_t\left(\bar{K}_t, \bar{L}; 0\right)^{-1} - \left(A_t Z_t\right)^{-1} q_0^{\frac{1}{\alpha}-1}\right)^{-1},$$

which is the result stated in text.

Appendix E: User cost of capital in extended model

We derive the formula for the user cost of capital for our extended model, and it corresponds to the formula stated in text.

Let $R(q)$ be the user cost of a machine of type q, and let this be associated with some

[8]Derivation of the above expression is cumbersome and has been automated in the Mathematica notebook *MPK_TCD.nb*.

dividend earned for having this machine for one period of length dt and renting it out for the duration of that period. The key condition is that the profit from such an activity must be zero after accounting for the cost of acquisition, dividend, and resell value of the machine at $t + dt$.

The costs of buying a machine at t and holding it for one period of length dt comprises its nominal purchase price at time t, which is $P_t k_t(q)$, and the opportunity cost of funds $\rho P_t k_t(q)\, dt$ incurred over period of length dt (ρ is the interest rate). The resell price is $P_{t+dt} k_{t+dt}(q)$, but since with assumed Poisson probability $\delta\Delta$ the machine disintegrates, the expected residual value is $(1-\delta) P_{t+dt} k_{t+dt}(q)\, dt$. The zero profit condition is thus given by

$$\underbrace{R_t(q)}_{\text{user cost}}\, dt = \underbrace{(1+\rho) P_t k_t(q)\, dt}_{\text{acquisition cost}} - \underbrace{(1-\delta) P_{t+dt} k_{t+dt}(q)\, dt}_{\text{residual value after a period of use}}.$$

Assuming balanced growth, assume Pk grows at rate $\gamma_{k,t} > 1$ from one period to the next (from t to $t + dt$). This simplifies the above expression to $R_t(q) = (1+\rho-(1-\delta)\gamma_{k,t}) P_t k_t(q)$. Given how we used r in the previous section, and assuming BGP, we obtain $r_t = (1+\rho-(1-\delta)\gamma_k) P_t$.

Appendix F: Continuity of Lebesgue integrals

We lack a good reference for this result and outline the proof here for completeness. The claim is that the function

$$g(x) := \int_a^x f(q)\, dv$$

is a continuous function; that is,

$$\lim_{x_n \to x_0} \int_a^{x_n} f(q)\, dv = \int_a^x f(q)\, dv,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue integrable function, $a \in \mathbb{R}$, and "$\int$" pertains to a Lebesgue integral (v is the Lebesgue measure). *Proof:* Define an indicator function $1_S(q)$ that takes the value of 1 on the sub-scripted set S and rewrite the left-hand side as

$$\lim_{x_n \to x_0} \int_a^{x_n} f(q)\, dv = \lim_{x_n \to x_0} \int_{-\infty}^{+\infty} 1_{[a,x_n]}(q) f(q)\, dv.$$

Since $|1_{[a,x_n]}(q) f(q)| \leq |f(q)|$, we have $1_{[a,x_n]}(q) f(q) \to f(q)$ (a.e.) on $[a, x_0]$ (see from * below for formal argument). Given $|f(q)|$ is Lebesgue integrable by assumption, we can use dominated convergence theorem (DCT) and enter with the limit under the integral, which yields

$$\lim_{x_n \to x_0} \int_a^{x_n} f(q)\, dv = \int_{-\infty}^{+\infty} \lim_{x_n \to x_0} 1_{[a,x_n]}(q) f(q)\, dv.$$

*)Observe that $\lim_{x_n \to x_0} 1_{[a,x_n]}(q) = 1_{[a,x_0]}(q)$ (a.e.), since for any $q \leq x_0$, we have $1_{[a,x_n]}(q) = 1$, and for any $q > x_0$, there exists an N such that for all $n \geq N$ we have $1_{[a,x_n]}(q) = 0$. The set on which $1_{[a,x_n]}(q)$ and $1_{[a,x_0]}(q)$ disagree is of the form $[x_0, b_n]$, where $b_n \to x_0$, and hence its Lebesgue measure is zero in the limit, as claimed. Consequently, by DCT, we have

$$\int_{-\infty}^{+\infty} \lim_{x_n \to x_0} 1_{[a,x_n]}(q) f(q)\, dv = \int_{-\infty}^{+\infty} 1_{[a,x_0]}(q) f(q)\, dv,$$

which finishes the proof.

Appendix G: Feasibility of microfoundations for CD task technology

Here we discuss additional example of mechanical random processes that could give rise to Pareto distributed capital requirements. As in the paper, the discussion draws on Newman (2004) and especially Gabaix (2009).

Random growth Random growth model is one of the simplest mechanisms to obtain power law dynamics. To see how it could work within our framework, suppose that capital requirement on average declines at a rate $\gamma < 1$ per unit of time. That is, $\bar{k}_{t+dt} = \gamma \bar{k}_t$, where $\bar{k}_t$ is the mean value across all tasks (time being discrete $dt > 0$ or continuous $dt \to 0$). Furthermore, assume the distribution of the decline is uneven across individual tasks because innovations affect individual tasks differently, and as in the paper deflate each variable by growth factor γ^t. As is clear from the setup, some tasks may not decline at all in a given period—in which case the relative capital requirement deflated by average growth factor γ^t is rising—while other tasks may decline by more than the average and so their deflated requirement is falling. The important assumption here is that this process is i.i.d. across tasks. The discussion of random growth model in Gabaix (2009) now applies, including the discussion of the variations of this model that can generate a power law tail index below 1 when this reasoning is directly applied to capital requirement $k(q)$.

Endogenous technology verse function The fact that capital requirement is an inverse of productivity of capital can be used to obtain tail power law from the basic fact of taking an inverse of diffused observation (Sornette, 2002). Let $y = x^{\frac{1}{\zeta-1}}, \zeta - 1 > 0$ and suppose x is distributed according to some pdf $p_x(x)$ such that $p(x) \to C > 0$ as $x \to 0$. Then, the tail distribution of y follows a power law with exponent ζ. Of course, applying this result requires that the economy operates far into the tail of the distribution, or else it will not even approximately behave as our CD task technology.[9]

Yule process It is also possible to employ Yule's "speciation" process. As in the case of the example discussed in the paper, the key to this approach to endogenize Pareto distribution is the observation that a variable that grows exponentially and is stopped after an exponentially distributed time is Pareto distributed at the stopping time.[10] This extension is fairly involved and we omit it from. However, based on the insights from information theory, it is possible to obtain a bridge between our model and the combinatorial growth literature (Weitzman, 1998; Jones, 2021) and show that the resulting distribution that involves "speciation" of ideas is Pareto. Preliminary results are available upon a request.

Appendix H: Corollary to Uzawa's theorem

We lack a good reference for this result and outline it here for completeness. The appendix shows how an additional assumption of declining price of capital goods leads to Cobb–Douglas production function.

[9]The result follows from the change of variables formula.

[10]The key mathematical property is that an exponential of an exponentially distributed random variable is Pareto distributed, as the following calculation shows ($X \sim$exponential, $Y = \exp(X)$):

$$Pr(Y \leq y) = Pr(\exp(X) \leq y) = Pr(X \leq \log(y)) = 1 - x^{-\lambda}.$$

By Theorem 2.6 and Theorem 2.7 in Acemoglu (2009), balanced growth path with positive and constant factor shares from $t \geq 0$ implies that one can find a sequence $\{a_t\}$ such that along that path $Y_t\left(K_t, \bar{L}\right) = Y_0\left(K_t, a_t\bar{L}\right)$ (Theorem 2.6) and $\frac{\partial Y_t\left(K_t,\bar{L}\right)}{\partial K_t} = \frac{\partial Y_0\left(K_t,a_t\bar{L}\right)}{\partial K_t}$, $\frac{\partial Y_t\left(K_t,\bar{L}\right)}{\partial \bar{L}} = \frac{\partial Y_0\left(K_t,a_t\bar{L}\right)}{\partial \bar{L}}$ (Theorem 2.7). Consider now the following definition: capital-augmenting progress occurs on the balanced growth path iff $k_t := K_t/a_t\bar{L}$ grows at a strictly positive rate. To see that this is a necessary and sufficient condition to imply that $Y_0\left(K_t, a_t\bar{L}\right)$ is CD, note we can express production along the balanced growth path as $f\left(k_t\right) := Y_0(k_t, 1)$, and that the constancy of the capital share implies $k_t f'\left(k_t\right)/f\left(k_t\right) = \alpha$ on that path, for some constant $0 < \alpha < 1$ and for all $t \geq 0$. Since k_t is growing and sweeps the entire domain (k_0, ∞), we obtain an ODE that solves to $f\left(k\right) = Ck^{\alpha}$ for some constant C, which yields the result: $Y_0\left(K, a\bar{L}\right) = CK^{\alpha}\left(a\bar{L}\right)^{1-\alpha}$. Concluding, CD production function obtains in any environment that restricts the balanced growth path to be such that k_t must grow over time, either by building it into the environment or requiring an equilibrium condition that implies that (e.g. a steadily declining price of capital goods).

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